Rational curves on del Pezzo surfaces in positive characteristic

Sho Tanimoto,
Nagoya University
joint work with R. Beheshti, B. Lehmann, and E. Riedl

ZORP, June 4th 2021
Caution: work in progress!!

Section 1

Introduction/Preliminaries

Del Pezzo surfaces

S: a weak del Pezzo surface over a finite field k,
i.e., S is a smooth projective surface s.t. $-K_S$ is big and nef.
Famous examples are smooth cubic surfaces in $\mathbb{P}^3$.

Manin's conjecture over finite fields asks how many rational curves of anticanonical degree $\leq d$ defined over k we have on S.

What does it mean?

Manin's conjecture over finite fields

$\mathrm{Mor}(\mathbb{P}^1, S)$: the scheme parametrizing morphisms from $\mathbb{P}^1$ to S.

Let $U \subset S$ be a Zariski open subset.

$$\mathrm{Mor}_U(\mathbb{P}^1, S) = \{f \in \mathrm{Mor}(\mathbb{P}^1, S) | f(\mathbb{P}^1) \cap U \neq \emptyset.\}$$

Let $\alpha \in N_1(S)$ be an integral effective 1-cycle.

$$\mathrm{Mor}_U(\mathbb{P}^1, S, \alpha) = \{f \in \mathrm{Mor}_U(\mathbb{P}^1, S) | f_*(\mathbb{P}^1) = \alpha\}$$

This is a scheme of finite type over k.

For any irreducible component $M \subset \mathrm{Mor}_U(\mathbb{P}^1, S, \alpha)$, one has

$$\dim M \geq -K_S.\alpha + 2$$

This lower bound is called as the expected dimension.

Manin's conjecture over finite fields

Let $\mathrm{Eff}_1(S) \subset N_1(S)$ be the cone of effective 1-cycles.

Here is the counting function we consider:

$$N(U, -K_S, d) = \sum_{\alpha \in \mathrm{Eff}_1(S)_{\mathbb{Z}}, -K_S.\alpha \leq d} \#(\mathrm{Mor}_U(\mathbb{P}^1, S, \alpha)(k))$$

In his unpublished notes from 1988, Batyrev developed a heuristic to obtain the asymptotic formula for this counting function.

Batyrev's heuristic

Let $\mathrm{Nef}_1(S) \subset N_1(X)$ be the nef cone of 1-cycles.

First Batyrev assumed that there exists a Zariski open subset U such that any rational curve meeting with U deforms to a free rational curve $f : C \to S$, i.e., f^*T_S is nef.
$\implies$ for any effecitve 1-cycle α which is not nef,

$$\mathrm{Mor}_U(\mathbb{P}^1, S, \alpha) = \emptyset$$

and any nef class α and any component $M \subset \mathrm{Mor}_U(\mathbb{P}^1, S, \alpha)$

$$\dim M = -K_S.\alpha + 2$$

Next Batyrev also assumed that for any $\alpha \in \mathrm{Nef}_1(S)_{\mathbb{Z}}$, the space

$$\mathrm{Mor}_U(\mathbb{P}^1, S, \alpha)$$

is irreducible.

Batyrev's heuristic

Finally Batyrev assumed that approximately we have

$$\#(\mathrm{Mor}_U(\mathbb{P}^1, S, \alpha)(k)) \sim q^{-K_S.\alpha+2}$$

(Later Ellenberg and Venkatesh suggested that one can achieve this by combining some homological stability results with Grothendieck trace formula.)

$$\begin{aligned} N(U, -K_S, d) &= \sum_{\alpha \in \mathrm{Eff}_1(S)_{\mathbb{Z}}, -K_S.\alpha \leq d} \#(\mathrm{Mor}_U(\mathbb{P}^1, S, \alpha)(k)) \\ &\sim \sum_{\alpha \in \mathrm{Nef}_1(S)_{\mathbb{Z}}, -K_S.\alpha \leq d} q^{-K_S.\alpha+2} \\ &\sim Cq^d d^{\rho(S)-1} \end{aligned}$$

where $C > 0$ and $\rho(S)$ is the Picard rank of S.

Moduli spaces of rational curves

$\overline{M}_{0,0}(S)$: the coarse moduli space of stable maps of genus 0 on S

In general it is difficult to understand this space
as some components may generically parameterize reducible curves.

Instead, we are interested in:

$\overline{\mathrm{Rat}}(S)$: the union of components of $\overline{M}_{0,0}(S)$ generically parametrizing stable maps from irreducible domains, i.e., $\mathbb{P}^1$

For any component $M \subset \overline{\mathrm{Rat}}(S)$, we have

$$\dim M \geq -K_S.C - 1 \quad (C \in M)$$

Moduli spaces of rational curves

What shall we ask about this moduli space?

- For each component M of $\overline{\mathrm{Rat}}(S)$, what is the dimension of M? Does it coincide with the expected dimension?
- What is the number of components parametrizing rational curves of anticanonical degree $\leq d$? Can we say something about the asymptotics?

Answers to these questions lead to applications on Manin's conjecture.

Testa addressed these questions for del Pezzo surfaces in characteristic 0.

Section 2

Fujita invariants

Fujita invariants

Definition (Fujita invariants)

X: a sm. proj. variety of dim. ≤ 3 over an alg. closed field k
L: a big and nef $\mathbb{Q}$-Cartier divisor on X
The Fujita invariant or a-invariant of (X, L) is

$$a(X, L) = \min\{t \in \mathbb{R} | tL + K_X \in \overline{\mathrm{Eff}}^1(X)\},$$

where $\overline{\mathrm{Eff}}^1(X)$ is the pseudo-effective cone of divisors.
By Das, $a(X, L) > 0 \iff \exists$ a dominant family of rat. curves C on X such that $K_X.C < 0$. When L is not big, we set $a(X, L) = +\infty$.
Even when X is singular, we define this invariant by pulling back to a smooth resolution $\beta : \widetilde{X} \to X$, i.e.,

$$a(X, L) = a(\widetilde{X}, \beta^* L).$$

This is well-defined because $a(X, L)$ is a birational invariant.

Fujita invariants

The *a*-invariants play a central role in Manin's conjecture:

Conjecture (Weak Manin's conjecture)

X: a uniruled smooth projective variety over a number field k.
L: a big and nef divisor on X.
$\mathrm{H}_L : X(k) \to \mathbb{R}_{\geq 0}$*: a height function associated to L*
Then $\exists$ a Zariski open subset $U \subset X$ such that
if we define counting function as

$$N(U, L, T) = \#\{P \in U(k) | \mathrm{H}_L(P) \leq T\},$$

then for any $\epsilon > 0$, we have

$$N(U, L, T) \ll T^{a(X,L)+\epsilon}$$

Fujita invariants

Theorem (Hacon–Jiang, '17)

X: a smooth uniruled projective variety over an algebraically closed field of characteristic 0
L: a big and nef $\mathbb{Q}$-divisor on X
V: the union of subvarieties Y such that $a(Y, L|_Y) > a(X, L)$.
Then V is a proper closed subset.

The proof used the boundedness of singular Fano varieties (BAB conjecture) proved by Birkar in an essential way.

Proposition

$f : Y \to X$: a generically finite dominant morphism between smooth projective varieties in characteristic 0.
L: a big and nef $\mathbb{Q}$-divisor on X.
Then $a(Y, f^*L) \leq a(X, L)$

Expected dimension of moduli space

Theorem (Lehmann-T, '19)

Assume the characteristic is 0.
X*: a smooth weak Fano variety, i.e.,* $-K_X$ *is big and nef*
V*: the union of* Y *s.t.* $a(Y, -K_X|_Y) > a(X, -K_X) = 1$

Any component $M \subset \overline{\mathrm{Rat}}(X)$ *parametrizing a non-dominant family of rational curves will parametrize rational curves contained in* V.

In particular, any component M *parametrizing rational curves outside of* V *is a dominant component generically parametrizing free rational curves and has dimension = expected dimension.*

Pathological Example 1

$k = \mathbb{F}_3$.

Consider the pencil of cubics

$$s(x^3 + y^3 + zy^2) + t(z^3 + xy^2).$$

One can show that a blow up of eight points of the base locus of this pencil is a weak del Pezzo surface S of degree 1.

Pathological Example 1

$\beta : \widetilde{S} \to S$: the blow up of the base point for $|-K_S|$.

Then $|-\beta^* K_S|$ defines a quasi-elliptic fibration, i.e., a fibration $\pi : \widetilde{S} \to B = \mathbb{P}^1$ such that a general fiber is a cuspidal rational curve.

To construct the component of $\overline{M}_{0,0}(S)$ parametrizing deformations of C, one needs to take a purely inseparable base change by the Frobenious $F : B' = \mathbb{P}^1 \to B = \mathbb{P}^1$.

Let $Y = \widetilde{S} \times_B B'$ and $\beta : \widetilde{Y} \to Y$ be a generic smoothing. We denote the fibration $\widetilde{Y} \to B'$ by ρ whose generic fiber is isomorphic to $\mathbb{P}^1_\eta$.

Pathological Example 1

Now we take the base change and work over $K = \mathbb{F}_3(t)$.

C_K: a general rational general fiber of ρ_K on $\widetilde{Y}_K$ over K.
Then we have $a(C_K, -K_{S_K}) = 2 > a(S_K, -K_{S_K}) = 1$
Also note that C_K is isomorphic to $\mathbb{P}^1_K$.
In particular, the asymptotic formula for C_K is cq^{2d}.

Thus we need to remove a Zariski dense set of rational points on S_K in order to obtain the desired growth rate for rational points on the generic fiber S_K. (That means $cq^d d^{\rho(S)-1}$)

So the exceptional set for weak Manin's can be Zariski dense.
Note that these rational points we must remove will be contained in a thin subset of points of S_K coming from $f_K : Y_K \to S_K$.

Pathological Example 2

We work over $\mathbb{F}_2$.
We will recall an example of a surface by Cascini and Tanaka.

Suppose we blow-up $\mathbb{P}^2$ at all seven $\mathbb{F}_2$-points.
We will obtain a weak del Pezzo surface S of degree 2.

[Cascini-Tanaka] shows that the (-2)-curves on S will be precisely the strict transforms of the seven $\mathbb{F}_2$-lines on $\mathbb{P}^2$.

[Cascini-Tanaka] shows that $|-K_S|$ defines a purely inseparable degree 2 map to $\mathbb{P}^2$.

This map factors through the anticanonical model S' of S which has seven A_1-singularities.

Pathological Example 2

Let $w^2 = f_4(x, y, z)$ be the defining equation of S' in $\mathbb{P}(1, 1, 1, 2)$ where f_4 is a homogenous polynomial of degree 4.

By the construction f_4 has coefficients in $\mathbb{F}_2$.
Consider a morphism

$$f : \mathbb{P}^2 \to S' : (s : t : u) \mapsto (x : y : z : w) = (s^2 : t^2 : u^2 : f_4(s, t, u)).$$

Then the Frobeneous map $F : \mathbb{P}^2 \to \mathbb{P}^2$ factors through f.
Since $-K_S$ is the pullback of $\mathcal{O}(1)$ under the map $S \to \mathbb{P}^2$,
we see that $-f^*K_S = \mathcal{O}(2)$.
Thus $a(Y, -f^*K_S) = \frac{3}{2}$ while $a(S, -K_S) = 1$.

Again working over $K = \mathbb{F}_2(t)$, the exceptional set for S_K must contains a Zariski dense subset of rational points $f(\mathbb{P}^2(K))$.

Classification of Breaking maps [BLRT21]

S: a weak del Pezzo surface of degree d
$f : Y \to S$: a dominant generically finite morphism such that $a(Y, -f^*K_S) > a(S, -K_S)$. We call this as a breaking map

Then we are in one of the following situations:

- (1) $\mathrm{char}(k) = 2, 3$, $d = 1$,
  the Iitaka dimension of $D = a(Y, -f^*K_S)(-f^*K_S) + K_Y$ is 1, and the pushforward of a general fiber of the Iitaka fibration for D is a curve C on S satisfying $-K_S \cdot C = 1$.
  In this case we have $a(Y, -f^*K_S) = 2$ and C is a member of a quasi elliptic fibration in $|-K_S|$.
- (2) $\mathrm{char}(k) = 2$, $d = 2$,
  f is birationally equivalent to a purely inseparable map of degree 2 from $\mathbb{P}^2$ to the anticanonical model of S.
  In this case we have $a(Y, -f^*K_S) = 3/2$ and $|-K_S|$ defines a inseparable map.

Classification of Breaking maps [BLRT21]

- (3) $\mathrm{char}(k) = 2$, $d = 1$,
  f is birationally equivalent to a purely inseparable double cover from the quadric cone Q to the anticanonical model of S.
  In this case we have $a(Y, -f^*K_S) = 2$
  and $|-2K_S|$ defines a inseparable map.
- (4) $\mathrm{char}(k) = 2$, $d = 1$,
  f is birationally equivalent to a non-separable map of degree 4 from $\mathbb{P}^2$ to the anticanonical model of S.
  In this case we have $a(Y, -f^*K_S) = 3/2$
  and $|-2K_S|$ defines a inseparable map. (!!)

When S is a del Pezzo surface then none of (1)-(4) can occur.

We found examples in all situations of (1)–(4).
There is a complete classification of surfaces satisfying (1) and (2) by Kawakami and Nagaoka.

Pathological families of rational curves

Proposition (BLRT21)

X: a smooth weak Fano variety defined over k with $L = -K_X$
$f : Y \to X$: a breaking map from a smooth projective variety Y.

Suppose $\exists$a dom. comp. M of $\overline{M}_{0,0}(Y)$ parameterizing a rational curves $g : C \to Y$ such that $g^*(a(Y, L)f^*L + K_Y).C = 0$. Then a family of rational curves $f \circ g : C \to X$ on X has dimension higher than expected dimension.

Since $f : C \to Y$ is a rational curve, we have

$$\dim\ M \geq -g^*K_Y + \dim\ Y - 3.$$

On the other hand since we have $a(Y, L) > 1$, we conclude

$$\dim\ M > -g^*f^*K_X + \dim\ X - 3.$$

□

Section 3

Low degree rational curves

Low degree rational curves–expected dimension

Theorem (BLRT21, !!)

S: a weak del Pezzo surface of degree d over k.
When $d = 2$, we assume S is not the following exception:

1. $\mathrm{char}(k) = 2$ *and* $|-K_S|$ *defines a purely inseparable generically finite map.*

When $d = 1$, we assume S is not the following exceptions

1. $\mathrm{char}(k) = 2, 3$ *and a general member of* $|-K_S|$ *is singular.*

M: a component of $\overline{M}_{0,0}(S)$ parametrizing a family of birational maps to curves C with $-K_S \cdot C \leq 3$.
Then M has the expected dimension unless C is a (-2)-curve.

Low degree rational curves–separability

Theorem (BLRT21)

S: a del Pezzo surface of degree d.
Assume that when $d = 1$, $\mathrm{char}(k) \geq 11$.
M: a component of $\overline{M}_{0,0}(S)$ parametrizing a dominant family of curves C with $-K_S \cdot C \leq 3$.
When $d = 3$, we assume S is not the following exception:

1. $\mathrm{char}(k) = 2$, *S is the Fermat cubic surface.*

When $d = 2$, we assume S is not one of the following exceptions:

1. $\mathrm{char}(k) = 3$, *S is the double cover of $\mathbb{P}^2$ ramified along the curve $zx^3 + xy^3 + yz^3$.*
2. $\mathrm{char}(k) = 2$, *S is a double cover of $\mathbb{P}^2$ defined by the equation $w^2 + wy^2 + g_4$.*
3. $\mathrm{char}(k) = 2$, *S is the blow-up of the Fermat cubic surface.*

Then M parametrizes a separable family of curves

Fermat cubic

$\mathrm{char}(k) = 2$, the Fermat cubic surface

$$S : x^3 + y^3 + z^3 + w^3 = 0$$

Rational curves in $|-K_S|$ are parametrized by the dual variety

$$S^* \subset (\mathbb{P}^3)^*$$

However, the Gauss map $S \to S^*$ is purely inseparable
and a general tangent plane cuts out a cuspidal rational curve C.

This means that if $f : C^\nu \to C \subset S$ be the normalization, then

$$N_{f/S} = \mathcal{O}(-1) \oplus k[t]/(t^2)$$

where the torsion part is supported on the cusp.

Thus $f : C^\nu \to S$ is not free.

a del Pezzo surface of degree 2

$\mathrm{char}(k) = 3$
S: the double cover of $\mathbb{P}^2$ ramified along the curve

$$D : zx^3 + xy^3 + yz^3.$$

Rational members of $|-K_S|$ is parametrized by the dual curve D^*.

Again the Gauss map $D \to D^*$ is purely inseparable,
and this means that every tangent line has a flex

$\implies$ a general member of $|-K_S|$ is cuspidal so the normal sheaf is

$$\mathcal{O}(-1) \oplus k(p)$$

where the torsion is supported on the cusp. Thus it is not free.

Some restrictions on $\mathrm{char}(k)$

$$\delta(d) = \begin{cases} 2 & \text{if } d \geq 4 \\ 3 & \text{if } d = 2, 3 \\ 11 & \text{if } d = 1. \end{cases}$$

Theorem (BLRT21)

S: a weak del Pezzo surface of degree d over k with $\mathrm{char}(k) = p$. Assume that $p \geq \delta(d)$. We assume that when $d = 2$ $p = 3$, S is a del Pezzo surface which is not listed in Theorem on separability. Then any dominant family of rational curves on S of anticanonical degree ≤ 3 contains a free rational curve.
In particular, any dominant component parametrizing rational curves of anticanonical degree ≤ 3 is separable so that it has expected dimension.

A proof is based on deformation theory of rational curves developed by Ito-Ito-Liedtke.

Section 4

Geometric Manin's conjecture

Bend and Break

Lemma (Bend and Break)

S: a weak del Pezzo surface over k. Fix a positive integer $d \geq 4$.

Assume every component of $\overline{\mathrm{Rat}}(S)$ parametrizing a dominant family of birational maps with degree $< d$ has expected dimension. $\overline{M} \subset \overline{\mathrm{Rat}}(S)$: a component parametrizing a dominant family of birational maps of degree d with $r = \dim M$

Fix $r - 1$ general points of S. Then $\exists$ a stable map $f : Z \to X$ whose image contains all $r - 1$ points will have two components $Z_1, Z_2 \subset Z$ such that $f|_{Z_1}$ and $f|_{Z_2}$ are general members of moduli in lower anticanonical degree.

If furthermore S is a smooth del Pezzo, then we can ensure that Z_1, Z_2 are the only components of Z.

Expected dimension

Proposition (BLRT21)

S: a weak del Pezzo surface over k.

Assume that every dominant component of $\overline{\mathrm{Rat}}(S)$ parametrizing rational curves on S of anticanonical degree ≤ 3 has the expected dimension.

$\overline{M} \subset \overline{\mathrm{Rat}}(S)$: any component parametrizing a dominant family of birational maps.

Then $\overline{M}$ has the expected dimension.

Expected dimension

We prove the statement by induction on the anticanonical degree. The base case when $-K_S \cdot C \leq 3$ is true by assumption.

Suppose that $-K_S \cdot C \geq 4$. Set $r = \dim(M)$.
By Bend-and-Break we find a stable map parametrized by M with reducible domain through $r-1$ general points of S.

By Lemma there are two curves C_1, C_2 in the image of f which deform in a dominant family. Since

$$r - 1 \leq (d_1 - 1) + (d_2 - 1) \leq -K_S \cdot C - 2 \leq r - 1$$

we see that $r = -K_S \cdot C - 1$.

$\square$

Expected dimension

Theorem (BLRT21, !!)

S: a weak del Pezzo surface of degree d over k.
$M \subset \overline{\mathrm{Rat}}(S)$: a dominant component

When $d = 2$, we assume S is not the following exception:

1. $\mathrm{char}(k) = 2$ *and* $|-K_S|$ *defines a purely inseparable generically finite map.*

When $d = 1$, we assume S is not the following exceptions

1. $\mathrm{char}(k) = 2, 3$ *and a general member of* $|-K_S|$ *is singular.*

Then M has the expected dimension.

Separability

Theorem (BLRT21)

S: a del Pezzo surface of degree d.
Assume that when $d = 1$, $\mathrm{char}(k) \geq 11$.
M: a dominant component of $\overline{\mathrm{Rat}}(S)$
When $d = 3$, we assume S is not the following exception:

1. *$\mathrm{char}(k) = 2$, S is the Fermat cubic surface.*

When $d = 2$, we assume S is not one of the following exceptions:

1. *$\mathrm{char}(k) = 3$, S is the double cover of $\mathbb{P}^2$ ramified along the curve $zx^3 + xy^3 + yz^3$.*
2. *$\mathrm{char}(k) = 2$, S is a double cover of $\mathbb{P}^2$ defined by the equation $w^2 + wy^2 + g_4$.*
3. *$\mathrm{char}(k) = 2$, S is the blow-up of the Fermat cubic surface.*

Then M parametrizes a separable family of curves

Separability

Let C be a general member of M and let $-K_S \cdot C = d$.
We prove our statement by induction on d.
By assumption the desired statement holds when $d \leq 3$.

When $d \geq 4$, we apply Bend-and-Break to find a stable map parametrized by M whose domain has exactly two components.

Furthermore Bend and Break guarantees that the images C_1, C_2 are general in their respective families, hence free.

Thus the restriction of the tangent bundle to this stable map is globally generated. We deduce that the general map parametrized by M is also free.

$\square$

Geometric Manin's conjecture

$\overline{M}^{bir}(S)$: the closure of the locus in $\overline{M}_{0,0}(S)$ parametrizing generically birational maps with irreducible domains.

Lemma (BLRT21)

S: a smooth del Pezzo surface of degree d defined over k, and $\beta \in N_1(S)_{\mathbb{Z}}$ with $e := -K_S \cdot \beta \geq 3$.
Assume that $p \geq \delta(d)$.
We assume when $d = 2$, S is not listed in Theorem on separability.
Let $q_1, \ldots, q_{e-2}$ be general points in S and let B be the locus in $\overline{M}^{bir}(S, \beta)$ parametrizing stable maps whose images pass through $q_1, \ldots, q_{e-2}$.
Then B is of dim. 1 and in the smooth locus of $\overline{M}_{0,0}(S, \beta)$.
$\exists$only finitely many maps in B have reducible domains.

A proof is by induction on e.
The base case is by Theorem using Ito-Ito-Liedtke

Geometric Manin's conjecture

Theorem (BLRT21)

S: a del Pezzo surface of degree d over k with characteristic p and let β be a nef curve class of anti-canonical degree $e \geq 3$.
We assume that $p \geq \delta(d)$.
We assume when $d = 2$, S is not listed in Theorem on separablity.
Then $\overline{M}^{bir}(S, \beta)$ is irreducible or empty.

Proof.

We lift everything to characteristic 0.
Using the specialization argument combined with Testa,
we conclude that B in the previous lemma is connected.
But lemma says that B is in the smooth locus of $\overline{M}_{0,0}(S)$
From this one can conclude $\overline{M}^{bir}(S, \beta)$ is irreducible.

□

Ongoing investigation

Up to now we observed the following theorem:

Theorem (BLRT21, !!)

S: a weak del Pezzo surface over k,

Then there is a dominant component of $\overline{\mathrm{Rat}}(S)$ whose dimension is higher than expected dimension if and only if there is a breaking map $f : Y \to S$, i.e., $a(Y, -f^*K_S) > a(S, -K_S) = 1$.

Question

Can we prove that all pathological families of rational curves with dimension higher than expected factor through a breaking map up to a base change of the family?

We are working on this problem using foliation theory.

Questions

The equation for Cascini–Tanaka's example has been obtained by Kawakami and Nagaoka, and it is given by

$$S : w^2 = xyz(x + y + z) \text{ in } \mathbb{P}(1, 1, 1, 2).$$

Question

Can we count rational points on S over $\mathbb{F}_2(t)$?
What's the universal torsor for S?

Thank you!!