Density of rational points on a family of del Pezzo surfaces of degree 1

Rosa Winter

MPI MIS Leipzig

Zoom On Rational Points

February 19th, 2021

Rational points on varieties

X a variety over a field k.

$X(k)$ set of k-rational points of X.

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Example

X a curve of genus at least 2, then $X(k)$ is finite.

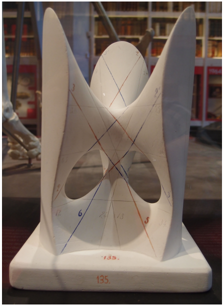
An example: cubic surfaces

Cubic surfaces

A class of del Pezzo surfaces: smooth cubic surfaces in $\mathbb{P}^3$.

Example

$x^3 + y^3 + z^3 + w^3 = (x + y + z + w)^3$ (Clebsch surface)



The geometry of cubic surfaces

Theorem (Cayley-Salmon, 1849)

- *A smooth cubic surface over an algebraically closed field contains exactly 27 lines.*
- *Any point on the surface is contained in at most three of those lines.*

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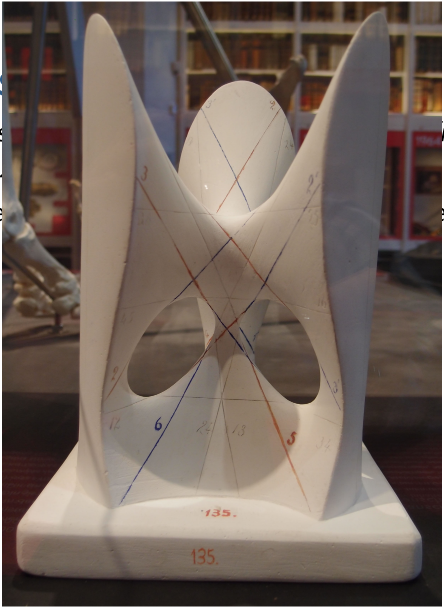

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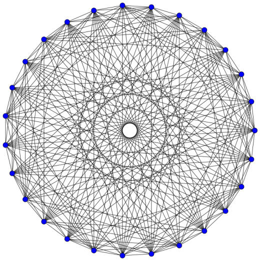

The intersection graph of the lines is the complement of the Schläfli graph.

Rational points on cubic surfaces

Let X be a smooth cubic surface over a field k.

Theorem (Segre, Manin, Kollár)

The following are equivalent.

(i) X contains a k-rational point.

(ii) There is a map $\mathbb{P}_k^n \dashrightarrow X$ for some n such that the image is dense in X.

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Property (ii) means that X is *unirational* over k. If k is infinite, this implies that $X(k)$ is dense in X.

Conclusion: k infinite, then

$$X(k) \neq \emptyset \text{ if and only if } X(k) \text{ dense in } X.$$

Del Pezzo surfaces

More general: del Pezzo surfaces

Definition

A *del Pezzo surface* X is a 'nice' surface with ample anticanonical divisor $-K_X$, i.e., X has an embedding in some $\mathbb{P}^n$, such that $-aK_X$ is linearly equivalent to a hyperplane section for some a.

Degree: self intersection $(-K_X)^2$ of the anticanonical divisor.

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Example

- Smooth cubic surfaces in $\mathbb{P}^3$ (degree 3).
- Complete intersection of two quadrics in $\mathbb{P}^4$ (degree 4).
- Double cover of $\mathbb{P}^2$, ramified over a smooth quartic curve (degree 2).
- For $3 \leq d \leq 9$, a del Pezzo surface is isomorphic to a surface of degree d in $\mathbb{P}^d$.

Geometry of del Pezzo surfaces

A del Pezzo surface over an algebraically closed field is birationally equivalent to the projective plane.

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Theorem

Let X be a del Pezzo surface of degree d over an algebraically closed field. Then X is isomorphic to either the product of two lines (then $d = 8$), or $\mathbb{P}^2$ blown up in $9 - d$ points in general position, where general position means

- *no three points on a line;*
- *no six points on a conic;*
- *no eight points on a cubic that is singular at one of them.*

Lines on a del Pezzo surface

Recall: a smooth cubic surface in $\mathbb{P}^3$ over an algebraically closed field contains 27 lines.

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Let X be a del Pezzo surface over an algebraically closed field, constructed by blowing up $\mathbb{P}^2$ in r points $P_1, \ldots, P_r$. There is a finite number of 'lines' (exceptional curves) on X. These are given by

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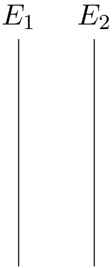
- the exceptional curves above $P_1, \ldots, P_r$;
  the strict transform of
- lines through two of the points;
- conics through five of the points;
- cubics through seven of the points, singular at one of them;
- quartics through eight of the points, singular at three of them;
- quintics through eight of the points, singular at six of them;
- sextics through eight of the points, singular at all of them, containing one of them as a triple point.

Degree 7

Blow up 2 points



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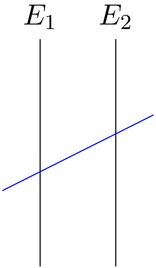

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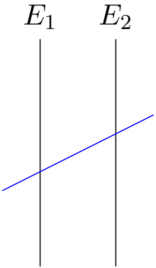
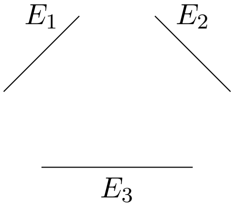


Degree 6

Blow up 3 points



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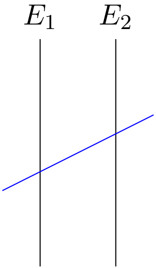

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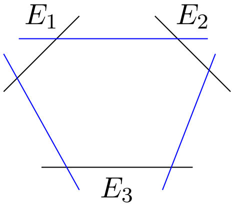

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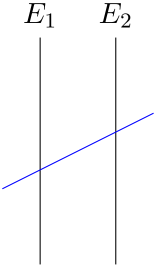

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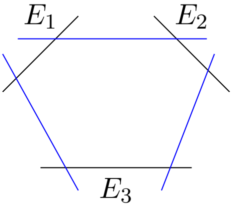
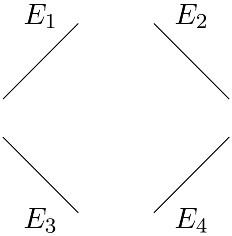


Degree 5

Blow up 4 points



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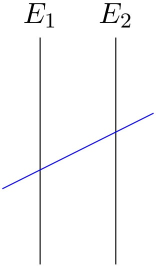

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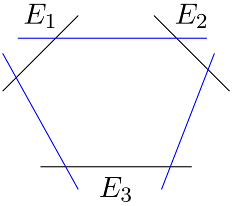

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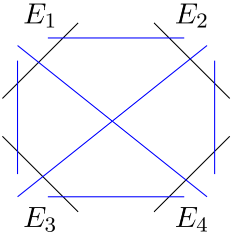

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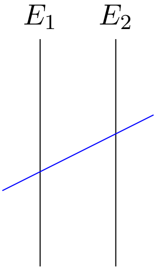

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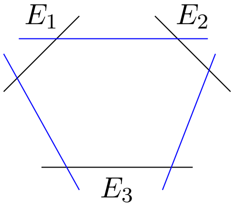

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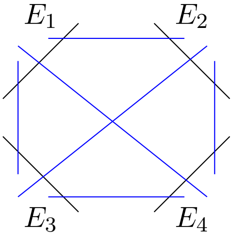

d	1	2	3	4	5	6	7	8
lines on X	240	56	27	16	10	6	3	1

Configurations of lines

The intersection graph of the lines on a del Pezzo surface is known:

Degree 5: 10 lines, Petersen graph.
Degree 4: 16 lines, Clebsch graph.
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Degree 2: 56 'lines', any line l intersects exactly one other line l' with multiplicity two, and 27 other lines with multiplicity one. These 27 lines do not intersect l', and they form again the complement of the Schläfli graph.
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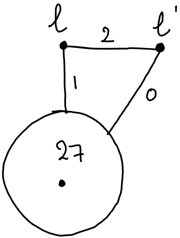

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Degree 1: 240 'lines'...

Theorem (van Luijk-W.)

On a del Pezzo surface of degree 1, a point is contained in at most 10 lines in char $\neq 2, 3$, at most 16 lines in char 2, and 12 in char 3.

Rational points on del Pezzo surfaces

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Recall:

• A variety X is *unirational* over a field k if there is a map $\mathbb{P}^n_k \dashrightarrow X$ for some n such that the image is dense in X.

• A smooth cubic surface has a k-rational point if and only if it is unirational over k.

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Theorem (Segre, Manin, Kollár, Pieropan)

A del Pezzo surface of degree $d \geq 3$ over a field k that has a k-rational point is unirational over k.

Rational points on del Pezzo surfaces

Del Pezzo surface of degree 2: double cover of $\mathbb{P}^2$, ramified over a smooth quartic curve.

Theorem (Salgado–Testa–Várilly-Alvarado)

A del Pezzo surface of degree 2 over a field k, that contains a k-rational point outside the ramification locus, that is not contained in the intersection of 4 lines, is unirational over k.

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Corollary

For a del Pezzo surface of degree $d \geq 2$ over an infinite field that contains a rational point (with extra condition if $d = 2$), the set of k-rational points is dense.

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• If X is not k-minimal (i.e. it contains a k-Galois orbit of pairwise disjoint exceptional curves), blow down to obtain a del Pezzo surface X' of higher degree.
• Use previous theorems do determine if $X'(k)$ is dense in X'.
• Since X and X' are birationally equivalent, density of $X(k)$ follows from density of $X'(k)$.

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X minimal $\Rightarrow$ X has Picard rank 1 or 2.

What about del Pezzo surfaces of degree 1?

Theorem (Kollár-Mella, 2017)

A del Pezzo surface of degree 1 over a field k with char $k \neq 2$ that admits a conic bundle structure is unirational.

The surfaces in the theorem have Picard rank 2.

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Apart from this theorem, the question is wide open:

Q. Is there an example of a minimal del Pezzo surface of degree 1 with Picard rank 1 that is unirational?

Q. Is there an example of a minimal del Pezzo surface of degree 1 with Picard rank 1 that is **not** unirational?

A goal more within reach

Let X be a variety, k an infinite field.

$$X \text{ unirational over } k \Rightarrow X(k) \text{ dense in } X.$$

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Q: X a del Pezzo surface of degree 1 over k. Is $X(k)$ dense in X with respect to the Zariski topology?

- We expect the answer to be yes.
- Partial results (Várilly-Alvarado '11, Ulas, Togbé, Jabara '07-'12, van Luijk–Salgado '14, Jardins-W.).

Summary

X a del Pezzo surface of degree d over an infinite field k. Assume there is a point $P \in X(k)$.

d	k-unirational ($\mathbb{P}^n \dashrightarrow X$ dominant)	$\Rightarrow$	Zariski density of $X(k) \subset X$
≥ 3	✓		✓
2	P outside a closed subset		P outside a closed subset
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Rest of this talk: explain the strategy in proving these partial results, and show new result (joint with Julie Desjardins).

Density of rational points on del Pezzo surfaces of degree 1

Del Pezzo surfaces of degree 1

A del Pezzo surface X of degree 1 over a field k is isomorphic to a smooth sextic in the weighted projective space $\mathbb{P}(2,3,1,1)$ with coordinates $(x:y:z:w)$:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

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Such a surface always contains a k-rational point :

$$\mathcal{O} = (1 : 1 : 0 : 0).$$

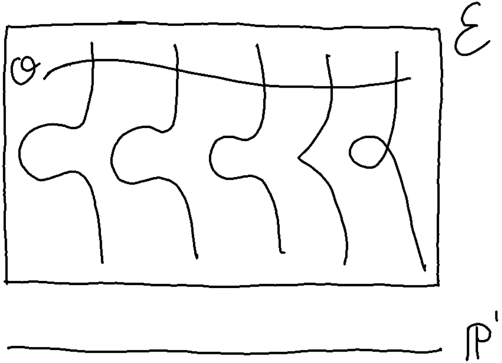
We have a map

$$X \dashrightarrow \mathbb{P}^1, \ \ (x : y : z : w) \longmapsto (z : w),$$

defined everywhere except in $\mathcal{O}$.

Del Pezzo surfaces of degree 1

When we blow up the point $\mathcal{O}$, we obtain an *elliptic surface*: a surface $\mathcal{E}$ with a morphism to $\mathbb{P}^1$, where almost all fibers are elliptic curves.



Example

Consider the del Pezzo surface given by

$$y^2 = x^3 + 27z^6 + 16w^6 \subset \mathbb{P}(2,3,1,1).$$

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Fiber above a point $(z_0 : w_0) \in \mathbb{P}^1$ is given by

$$y^2 = x^3 + 27z_0^6 + 16w_0^6,$$

isomorphic to an elliptic curve for allmost all $(z_0 : w_0)$.

Strategy to prove density of rational points

Recap:

X del Pezzo surface of degree 1 over a field k.

- X given by a smooth sextic in $\mathbb{P}(2,3,1,1)$.
- X contains a rational point $\mathcal{O}$.
- After blowing up $\mathcal{O}$, obtain an elliptic surface $\mathcal{E}$ with a dominant morphism $\mathcal{E} \longrightarrow X$.

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• After blowing up $\mathcal{O}$, obtain an elliptic surface $\mathcal{E}$ with a dominant morphism $\mathcal{E} \longrightarrow X$.

If $\mathcal{E}(k)$ is dense in $\mathcal{E}$, then $X(k)$ is dense in X.

Goal: prove density of $\mathcal{E}(k)$.

Showing density of $\mathcal{E}(k) \subset \mathcal{E}$ - two techniques

1. Studying the *root number* of the fibers:

Theorem (Várilly-Alvarado,'11)

Let X be a del Pezzo surface given by

$$y^2 = x^3 + Az^6 + Bw^6,$$

with $A, B \in \mathbb{Z}$, such that either $3A/B$ is not a square, or gcd$(A, B) = 1$ and $9 \nmid AB$. If the Tate-Shafarevich group of elliptic curves with j-invariant 0 is finite, then $X(\mathbb{Q})$ is dense in X.

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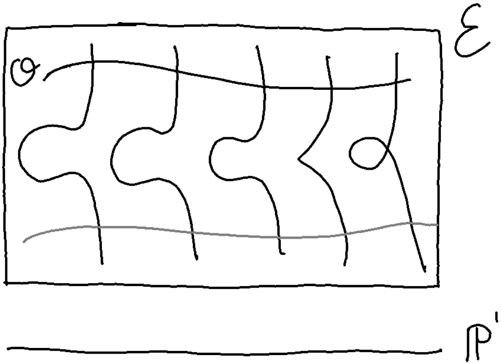
$\rightarrow$ Várilly-Alvarado showed that there are infinitely many disjoint pairs of fibers with opposite root number.

Showing density of $\mathcal{E}(k) \subset \mathcal{E}$ - two techniques

2. Creating a *multisection*:



Section

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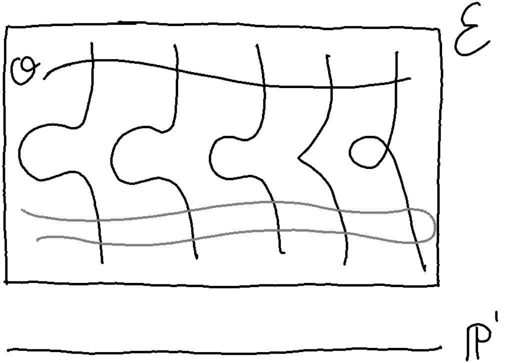

Multisection

Showing density of $\mathcal{E}(k) \subset \mathcal{E}$ - two techniques

2. Creating a *multisection*:

Ulas, Togbé, Jabara '07-'12, van Luijk–Salgado '14.

Results for various families of del Pezzo surfaces of degree 1, by creating multisections of genus ≤ 1 and assuming that they contain infinitely many rational points.

Recent result

Let X be a del Pezzo surface of degree 1 over an infinite field k of characteristic 0 of the form

$$y^2 = x^3 + az^6 + bz^3w^3 + cw^6, \tag{1}$$

with a, c nonzero. Let $\mathcal{E}$ be the elliptic surface obtained by blowing up the point $(1 : 1 : 0 : 0)$.

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Theorem (Desjardins–W.)

If X contains $P = (x_0 : y_0 : z_0 : w_0) \in X(k)$ with $z_0, w_0 \neq 0$, and its corresponding point on $\mathcal{E}$ lies on a smooth fiber and is non-torsion, then $X(k)$ is dense in X. If k is finitely generated over $\mathbb{Q}$, the converse holds as well.

Recent result

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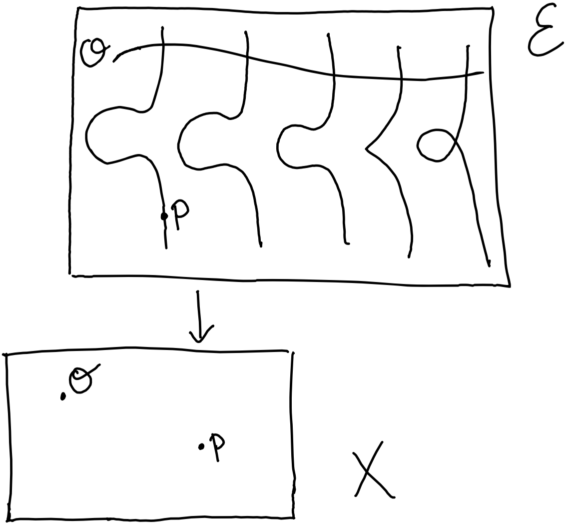
over $\mathbb{Q}$, the converse



Recent result

Let X be a del Pezzo surface of degree 1 over an infinite field k of characteristic 0 of the form

$$y^2 = x^3 + az^6 + bz^3w^3 + cw^6, \tag{1}$$

with a, c nonzero. Let $\mathcal{E}$ be the elliptic surface obtained by blowing up the point $(1 : 1 : 0 : 0)$.

Theorem (Desjardins–W.)

If X contains $P = (x_0 : y_0 : z_0 : w_0) \in X(k)$ with $z_0, w_0 \neq 0$, and its corresponding point on $\mathcal{E}$ lies on a smooth fiber and is non-torsion, then $X(k)$ is dense in X. If k is finitely generated over $\mathbb{Q}$, the converse holds as well.

This is the first result that gives sufficient **and** necessary conditions for the family (1) over any field finitely generated over $\mathbb{Q}$.

Proving the theorem

A family of 3-sections

$X,\ k,\ \mathcal{E}$ as in the theorem.

For $R = (x_R : y_R : z_R : w_R) \in \mathcal{E}$, with $z_R, w_R \neq 0$, we construct a 3-section C_R of $\mathcal{E}$:

C_R is the pullback to $\mathcal{E}$ of the intersection of X with

$$3x_R^2 z_R^2 xz - 2y_R z_R^3 y - (x_R^3 - 2az_R^6 - bz_R^3)z^3 + (2cz_R^3 + bz_R^6)w^3 = 0.$$

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We found this by looking at linear systems on X:

The linear system $|-nK_X|$ has dimension $\frac{1}{2}n(n+1)$ and a general curve in here has genus $\frac{1}{2}n(n-1)+1$.

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We found this by looking at linear systems on X:

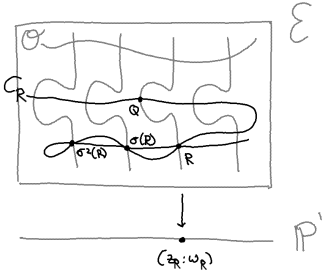
The linear system $|-3K_X|$ has dimension 6 and a general curve in here has arithmetic genus 4.

We found this curve in the linear system $|-3K_X - 2R|$.

Studying C_R closer

- Automorphism $\sigma\colon (x:y:z:w) \longmapsto (x:y:z:\zeta_3^2 w)$
- C_R is a 3-section, and singular in R, $\sigma(R)$, $\sigma^2(R)$.

Studying C_R closer



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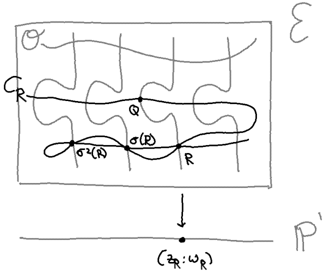
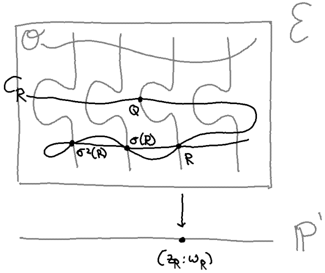


Proposition

If R is not contained in the pullback of an exceptional curve on $\overline{X} = X \times_k \overline{k}$, then C_R either contains a section that is defined over k, or it is geometrically integral and has geometric genus at most 1, in which case R, $\sigma(R)$, $\sigma^2(R)$ are all double points.

Studying C_R closer



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These are 240 sections of $\mathcal{E}$.

Recall: we assume there is $P = (x_0 : y_0 : z_0 : w_0) \in X(k)$ such that $z_0, w_0 \neq 0$, and its corresponding point on $\mathcal{E}$ lies on a smooth fiber $\mathcal{F}$, and is non-torsion on $\mathcal{F}$.

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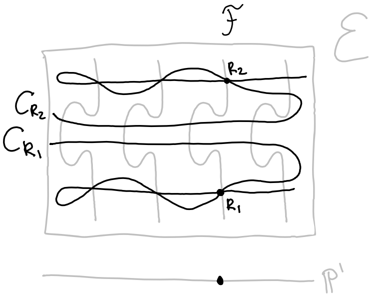
Corollary

There are infinitely many multiples R of P on $\mathcal{F}$ such that C_R either contains a section defined over k, or C_R has genus at most 1.



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Corollary

There are infinitely many multiples R of P on $\mathcal{F}$ such that C_R either contains a section defined over k, or C_R has genus at most 1.

• If there is an R such that C_R contains a section s over k:
$\rightarrow s$ has infinite order (MW group of $\mathcal{E}$ torsion-free).
$\rightarrow \infty$ many multiples of s on $\mathcal{E}$, each with ∞ many rational points.
$\rightarrow \mathcal{E}(k)$ dense in $\mathcal{E}$.

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Corollary

There are infinitely many multiples R of P on $\mathcal{F}$ such that C_R either contains a section defined over k, or C_R has genus at most 1.

- If there is an R such that C_R contains a section s over k: done

- If there is an R such that C_R has geometric genus 0:

$\rightarrow C_R$ has infinitely many k-rational points.
$\rightarrow$ Base change $\mathcal{E} \times_{\mathbb{P}^1} \tilde{C}_R \longrightarrow \tilde{C}_R$ has a section of infinite order with infinitely many rational points, hence $(\mathcal{E} \times_{\mathbb{P}^1} \tilde{C}_R)(k)$ dense, hence $\mathcal{E}(k)$ dense.

Recall: we assume there is $P = (x_0 : y_0 : z_0 : w_0) \in X(k)$ such that $z_0, w_0 \neq 0$, and its corresponding point on $\mathcal{E}$ lies on a smooth fiber $\mathcal{F}$, and is non-torsion on $\mathcal{F}$.

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There are infinitely many multiples R of P on $\mathcal{F}$ such that C_R either contains a section defined over k, or C_R has genus at most 1.

• If there is an R such that C_R contains a section s over k: done

• If there is an R such that C_R has geometric genus 0: done

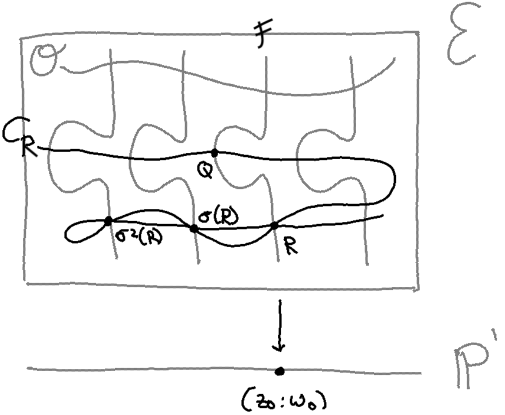
• What if none of the above occurs? $\rightarrow$ There are infinitely many $R \in \mathcal{F}$ such that C_R has geometric genus 1.

Let $\mathcal{F}_0 \subset \mathcal{F}$ be the set of these R.

Another elliptic fibration

For $R \in \mathcal{F}_0$ we have:

$E_R = (\tilde{C}_R, Q)$ elliptic curve, with point $D_R = \sigma(Q) + \sigma^2(Q)$.



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$\eta = (\tilde{x} : \tilde{y} : z_0 : w_0)$ generic point of $\mathcal{F}$ over the function field $k(\mathcal{F}) = k(\tilde{x}, \tilde{y})$ of $\mathcal{F}$.

- $C_\eta \subset \mathbb{P}_{k(\mathcal{F})}(2,3,1,1)$ the 3-section of $\mathcal{E} \times_k k(\mathcal{F})$
- E_η the corresponding elliptic curve over $k(\mathcal{F})$ with point D_η.

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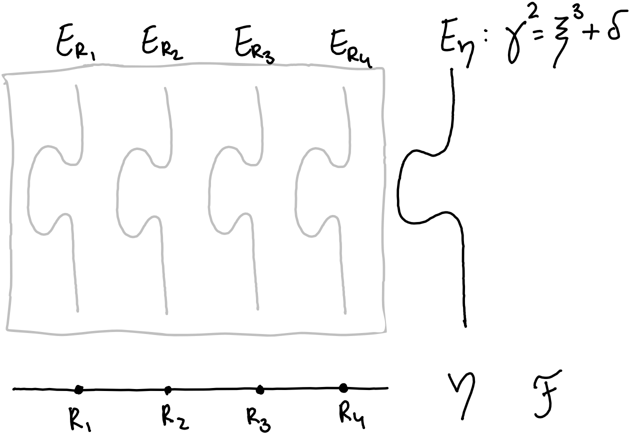
- E_η the corresponding elliptic curve over $k(\mathcal{F})$ with point D_η.

Proposition

D_η is non-torsion on E_η.

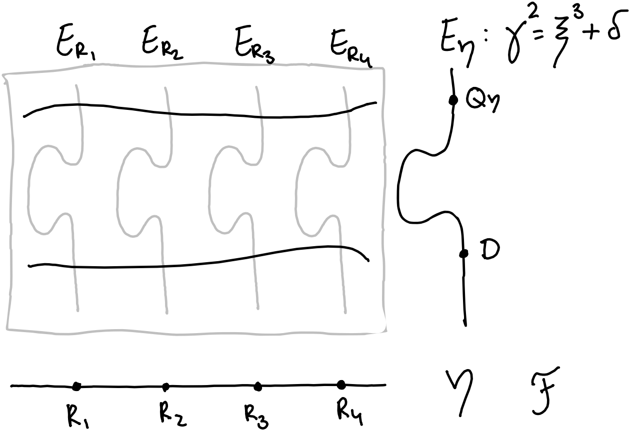
Another elliptic fibration

$E_\eta/k(\mathcal{F})$ gives rise to an elliptic surface $\mathcal{C}$ over $\mathcal{F}$...



Another elliptic fibration

...with a section of infinite order!



Finishing the argument

Conclusion: The elliptic surface $\mathcal{C}$ over $\mathcal{F}$ contains a section of infinite order, so the set of rational points $\mathcal{C}(k)$ is dense in $\mathcal{C}$.

Infinitely many fibers of $\mathcal{C}$ are birationally equivalent to 3-sections of $\mathcal{E}$, so $\mathcal{C}$ maps dominantly to $\mathcal{E}$.

So $\mathcal{E}(k)$ is dense in $\mathcal{E}$!

Recap

Corollary

There are infinitely many multiples R of P on $\mathcal{F}$ such that C_R either contains a section defined over k, or C_R has genus at most 1.

- If there is an R such that C_R contains a section s over k: done
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- What if none of the above occurs? $\rightarrow$ There are infinitely many $R \in \mathcal{F}$ such that C_R has geometric genus 1: done

Recap

Corollary

There are infinitely many multiples R of P on $\mathcal{F}$ such that C_R either contains a section defined over k, or C_R has genus at most 1.

So we showed that if X contains a point P as in the theorem, the set of rational points $X(k)$ is dense in X.

Conversely: k finitely generated over $\mathbb{Q}$. If $X(k)$ dense in X, then X contains a point P with infinite order on its fiber on $\mathcal{E}$; otherwise $X(k)$ would be contained in the torsion locus on $\mathcal{E}$, which is a closed subset (using a generalization of Merel, showed to us by Colliot-Thélène).

Example

Let X be the del Pezzo surface of degree 1 given by

$$y^2 = x^3 + 6(27z^6 + w^6).$$

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So X contains a point that lies on a smooth fiber of $\mathcal{E}$ and has infinte order, hence $X(\mathbb{Q})$ is dense in X!

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Explicitly:
• Two generators for $\mathcal{E}_{(1:1)}(\mathbb{Q})$ are given by $P_1 = (1 : 13 : 1 : 1)$ and $P_2 = (22 : 104 : 1 : 1)$ (magma).
• The curve C_{P_1} is cut out from X by $3xz - 26y + 323z^3 + 12w^3$.
• $E_{P_1} : \gamma^2 = \xi^3 - 2 \cdot 34 \cdot 52 \cdot 28368481$, point D_{P_1} on it has ξ-coordinate

$$\xi_D = \frac{11 \cdot 33487 \cdot 580020724757}{(2 \cdot 12 \cdot 167 \cdot 523)^2}.$$

Thank you!