Manin conjecture for algebraic stacks

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Nortcott property: for every $B > 0$ there are only finitely isomorphism classes of elliptic curves such that $H(E) < B$. ($H = H_{\text{naive}}, H = H_{\text{Falt}}$)

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Theorem (Hortsch. '15)

One has that

$$|\{E/\mathbb{Q} | H_{\mathrm{Falt}}(E) < B\}| = \frac{12\sigma}{\zeta(10)} B^{5/6} + O(B^{\frac{1}{2}}(\log(B))^3),$$

where

$$\sigma = \frac{2}{5}\int_{-\infty}^{\infty} \left| \frac{\Delta(\tau_t)\Im(\tau_t)^6}{16(4t+27)} \right|^{5/6} dt \quad (j(\tau_t) = \frac{6912t}{4t+27}).$$

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Let G be an algebraic group acting on a variety X. The stack X/G classifies G-torsors:

$$(X/G)(V) := \{G_V - \text{equivariant morphisms}\, T \to X_V, \\ \text{where}\, T \to V \text{ is a } G - \text{torsor}\}.$$

We write BG for the stack $\{\cdot\}/G$.

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When $n = 1, a \geq 1$, one has

$$B\mu_a = \mathcal{P}(a).$$

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Topology on $\mathcal{P}(\mathbf{a})(\mathbb{Q}_p)$

We say that $U \subset \mathcal{P}(\mathbf{a})(\mathbb{Q}_p)$ is open if for every morphism $f : V \to \mathcal{P}(\mathbf{a})$, with V locally of finite type scheme, the preimage $f(\mathbb{Q}_p)^{-1}(U)$ is open.

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$\mathrm{Pic}(\mathcal{P}(\mathbf{a})) = \mathbb{Z}$, if $n > 1$ and $\mathrm{Pic}(\mathcal{P}(\mathbf{a})) = \mathbb{Z}/a\mathbb{Z}$ if $n = 1$.

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$$\frac{|\widetilde{s}(t\cdot \mathbf{x})|_p}{||\widetilde{s}||_p^*(t\cdot \mathbf{x})} = |t|_p^k \frac{|\widetilde{s}(\mathbf{x})|_p}{||\widetilde{s}(\mathbf{x})||_p^*}.$$

A metric on $\mathcal{O}(k)(\mathbb{Q}_p) \to \mathcal{P}(4,6)(\mathbb{Q}_p)$ corresponds to a continuous function $f_p : Q_p^2 - \{0\} \to \mathbb{R}_{>0}$ s.t. $f_p(t\cdot\mathbf{x}) = |t|_p^k f_p(\mathbf{x})$.

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Problem:
Stable heights do not satisfy Northcott property. Two elliptic curves having the same j-invariant are not necessary isomorphic over $\mathbb{Q}$.

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Definition

Let $\mathbf{x} \in \mathcal{P}(4,6)(\mathbb{Q}_p)$ *and let* $\ell \in \mathcal{O}(12)(\mathbf{x})$. *We define*

$$||\ell||_p^{\#} = \inf_{\overline{\mathbf{x}}}\{ \inf_{a \in \mathbb{Q}_p^{\times}} \{|a|_p \big| \ell \in a\overline{\mathbf{x}}^{*}\mathcal{O}(12)\}\}.$$

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$$f_p^{\#}(\mathbf{x}) = p^{12r_p(\mathbf{x})}, \qquad r_p(\mathbf{x}) = \sup_{j=1,2} \left\lceil \frac{-v(x_j)}{a_j} \right\rceil.$$

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Definition

Consider family $(f_p)_p$ such that for almost all p one has $f_p = f_p^{\#}$. For $\mathbf{x} \in \mathcal{P}(4,6)(\mathbb{Q})$, pick a lift $\widetilde{\mathbf{x}} \in \mathbb{Q}^2 - \{0\}$.

Toric heights

The metric corresponds to $f_p^{\#} : \mathbb{Q}_p^2 - \{0\} \to \mathbb{R}_{>0}$ is given by

$$f_p^{\#}(\mathbf{x}) = p^{12 r_p(\mathbf{x})}, \qquad r_p(\mathbf{x}) = \sup_{j=1,2} \left\lceil \frac{-v(x_j)}{a_j} \right\rceil .$$

$$f_p^{\#}|_{\mathcal{D}_p^{(4,6)}} = 1 \qquad f_p(t \cdot \mathbf{x}) = |\pi_v|_v^{-12v(t)} f_p(\mathbf{x}).$$

$$f_\infty^{\#}(\mathbf{x}) = \max(|x_1|^2, |x_2|^3)$$

Definition

Consider family $(f_p)_p$ such that for almost all p one has $f_p = f_p^{\#}$. For $\mathbf{x} \in \mathcal{P}(4,6)(\mathbb{Q})$, pick a lift $\widetilde{\mathbf{x}} \in \mathbb{Q}^2 - \{0\}$.

$$H(\mathbf{x}) = \prod_p f_p(\widetilde{\mathbf{x}}).$$

Examples of toric height

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$$x \in \mathcal{P}(a)(\mathbb{Q}) = \mathbb{Q}^\times / \mathbb{Q}^{\times a} \implies H^{\#}(x) = \widetilde{x}, \quad (\widetilde{x} \in \mathbb{Z}, \forall p : p^a \not| \widetilde{x}).$$

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Faltings height

$p > 3$, p prime: precisely the toric metric, because it admits minimal model $Y^2 = X^3 + aX + b$ with $(a, b) \in \mathcal{D}_p^{(4,6)}$.

$p = 2, 3$: we need to change the metric.

$p = \infty$: $||Y^2||([a, b]) = |g_6^2(\tau)\Im(\tau)|^6 =_{\Im(\tau)\to+\infty} O(\log(|j(\tau)|)^6)$.

Proposition

There exist $C, m > 0$ such that

$$H_{\mathrm{Sing}}(\mathbf{x}) \geq CH^{\#}(\mathbf{x}) \log(1 + H^{\#}(\mathbf{x}))^{-m}.$$

Measures on $\mathcal{P}(4,6)(\mathbb{Q}_p)$

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If f_p is "singular", but f_p^{-1} is integrable over $\mathcal{D}_p^{(4,6)}$ if p finite and f_p^{-1} is integrable over $(\mathbb{Q}_p^\times)^{n-1} \times \{1\}$ we can still define measures.

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C_{\text{Peyre}}(\mathcal{P}(4,6)) :& \\
&= \prod_{p \text{ prime}} \left(\zeta_p(1)^{-1} \omega_p(\mathcal{P}(4,6)(\mathbb{Q}_p)) \right) \omega_\infty(\mathcal{P}(4,6)(\mathbb{R})) \\
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Theorem (D.)

Let H be a nonsingular toric height coming from $\mathcal{O}(a_1 + \cdots + a_n)$. There exists an open substack $\mathcal{U} \subset \overline{\mathcal{P}(4,6)}$ such that

$$\{\mathbf{x} \in \mathcal{U}(F) | H(\mathbf{x}) \leq B\} \sim C_{\text{Peyre}}(\overline{\mathcal{P}(\mathbf{a})})B.$$

Idea of the proof

Count rational points on the "stacky" torus

$$\mathcal{T}^{(4,6)} = \mathbb{G}_m^2/\mathbb{G}_m \subset \mathbb{A}^2 - \{0\}/\mathbb{G}_m = \mathcal{P}(\mathbf{a}) \subset \overline{\mathcal{P}(\mathbf{a})}.$$

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Let $H_p(\mathbf{s}, \mathbf{x}) = |x_1|_p^{-s_1}|x_2|_p^{-s_2} f_p^{\#}(\mathbf{x})^{2s_1/5+3s_2/5}$, we have $H(\mathbf{x}) = \prod_p H_p(1, 1, \mathbf{x})$.

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$$Z(\mathbf{s}) = \sum_{\mathbf{x}\in\mathcal{T}^{4,6}(\mathbb{Q})} H(-\mathbf{s},\mathbf{x}) = \sum_{\mathbf{x}\in i(\mathcal{T}^{4,6}(\mathbb{Q}))} H(-\mathbf{s},\mathbf{x})$$
$$= \int_{(\mathcal{T}^{4,6}(\mathbf{A}_{\mathbb{Q}})/i(\mathcal{T}^{4,6}(\mathbb{Q})))^*} \widehat{H}(\mathbf{s},\chi)d\chi^*$$

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There exists $C(G,F) > 0$ such that

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when $B \to \infty$, with $a(G)$ and $b(F,G)$ explicit.

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A Galois extension K/F of Galois group G defines a G-torsor $\mathrm{Spec}(K) \to \mathrm{Spec}(F)$ i.e. a rational point of BG.

Counting torsors

$G = \mu_a$, $a \geq 2$

Proposition (D.)

One has that

$$|\{x \in B\mu_a(F) | \mathcal{D}(\mathbf{x}) < B\}| \sim c(a,F) B^{\frac{1}{a-a/r}} (\log(B))^{r-2},$$

where r is the smallest prime of a.

Thank you!