An example of Brauer–Manin obstruction to weak approximation from a prime with good reduction

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Introduction

Let k be a number field and Ω_k be the set of places of k. Let X be a smooth, proper, geometrically integral k-variety. We are interested in understanding the set $X(k)$ of k-points on X.
For every $\nu \in \Omega_k$ we have

$$X(k) \hookrightarrow X(k_\nu).$$

Hence,

$$X(k) \hookrightarrow \prod_{\nu \in \Omega_k} X(k_\nu).$$

Question: what does the image of $X(k)$ in $\prod_{\nu \in \Omega_k} X(k_\nu)$ look like?

Weak approximation

Definition

We say that X satisfies weak approximation if the image

$$X(k) \hookrightarrow \prod_{\nu \in \Omega_k} X(k_\nu)$$

is dense.

(Counter)example

1. The projective spaces $\mathbb{P}^n_k$ satisfy weak approximation.
2. $Q \subseteq \mathbb{P}^n_k$ smooth projective quadric satisfies weak approximation.
3. Selmer's example: the projective variety defined by the equation $3x^3 + 4y^3 + 5z^3 = 0$ has a $\mathbb{Q}_\nu$-point for every place $\nu \in \Omega_{\mathbb{Q}}$ but does not admit a rational point.

Manin, 1970

Manin has shown that it is possible to use the Brauer group to build a closed subset $\left(\prod_{\nu\in\Omega_k} X(k_\nu)\right)^{\mathrm{Br}} \subseteq \prod_{\nu\in\Omega_k} X(k_\nu)$ that contains $X(k)$. Hence,

$$X(k) \subseteq \overline{X(k)} \subseteq \left(\prod_{\nu\in\Omega_k} X(k_\nu)\right)^{\mathrm{Br}} \subseteq \prod_{\nu\in\Omega_k} X(k_\nu).$$

Idea: $\left(\prod_{\nu\in\Omega_k} X(k_\nu)\right)^{\mathrm{Br}}$ is easier to describe then $\overline{X(k)}$.

Brauer–Manin obstruction to weak approximation

We say that there is a Brauer–Manin obstruction to weak approximation on X if $\left(\prod_{\nu\in\Omega_k} X(k_\nu)\right)^{\mathrm{Br}} \subsetneq \prod_{\nu\in\Omega_k} X(k_\nu)$.

The evaluation map

Let $\mathcal{A} \in \mathrm{Br}(X) := \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$, then for every place $\nu \in \Omega_k$ we get an induced map, called the evaluation map

$$\mathrm{ev}_{\mathcal{A}} : X(k_\nu) \to \mathrm{Br}(k_\nu)$$

Places involved in the obstruction

We say that ν plays a role in the Brauer–Manin obstruction to weak approximation if there exists $\mathcal{A} \in \mathrm{Br}(X)$ such that

$$\mathrm{ev}_{\mathcal{A}} : X(k_\nu) \to \mathrm{Br}(k_\nu)$$

is non-constant.

Swinnerton-Dyer's question

Let X be such that $\mathrm{Pic}(X \times_k \overline{k})$ is torsion-free and finitely generated. Let $S \subseteq \Omega_k$ be a finite set of places containing the archimedean places and the places of bad reduction for X.
Question: is it true that the only places that can play a role in the Brauer–Manin obstruction to weak approximation are contained in S?

The answer of Bright and Newton

Good ordinary varieties

Let k be a perfect field of characteristic p and Y a smooth, proper k-variety. Let $B_Y^q := \mathrm{im}(d : \Omega_Y^{q-1} \to \Omega_Y^q)$. We say that Y is ordinary if $\mathrm{H}^n(Y, B_Y^q) = 0$ for every n, q.

Theorem [Bright-Newton 2020]

Let X be a smooth, proper variety over a number field k with $\mathrm{H}^0(X, \Omega_X^2) \neq 0$. Let $\mathfrak{p}$ be a prime at which X has good ordinary reduction, with residue characteristic p. Then there exist a finite extension L/k, a prime $\mathfrak{p}'$ lying over $\mathfrak{p}$, and an element $\mathcal{A} \in \mathrm{Br}(X_L)\{p\}$ such that the evaluation map

$$\mathrm{ev}_{\mathcal{A}} : X(L_{\mathfrak{p}'}) \to \mathrm{Br}(L_{\mathfrak{p}'})$$

is non-constant.

Definition

A K3 surface over a number field k is a smooth, geometrically integral, 2-dimensional k-variety such that

$$\mathrm{H}^1(X, \mathcal{O}_X) = 0 \quad \text{and} \quad \omega_X \simeq \mathcal{O}_X$$

- The Picard group $\mathrm{Pic}(X \times_k \overline{k})$ is always torsion-free and finitely generated.
- $\mathrm{H}^0(X, \Omega^2_X) \neq 0$.
- If $\mathfrak{p}$ is a place of good reduction satisfying $e_{\mathfrak{p}} < p - 1$, then $\mathrm{ev}_{\mathcal{A}} : X(k_{\mathfrak{p}}) \to \mathrm{Br}(k_{\mathfrak{p}})$ is constant (Bright–Newton).

Remark

If we start with a K3 surface defined over the rational numbers, the only place with good reduction that can play a role in the Brauer–Manin obstruction to weak approximation is 2.

Example

Let $X \subseteq \mathbb{P}^3_{\mathbb{Q}}$ be the K3 surface defined by the equation

$$x^3y + y^3z + z^3w + w^3x + xyzw = 0. \qquad (1)$$

X has good reduction at the prime 2 and the following theorem holds true.

Theorem

The class of the quaternion algebra

$$\mathcal{A} = \left(\frac{z^3 + w^2x + xyz}{x^3}, -\frac{z}{x} \right) \in \mathrm{Br}(k(X))$$

defines an element in $\mathrm{Br}(X)$. The evaluation map $\mathrm{ev}_{\mathcal{A}} : X(\mathbb{Q}_2) \to \mathrm{Br}(\mathbb{Q}_2)$ is non-constant. Finally, $X(\mathbb{Q})$ is not dense in $X(\mathbb{Q}_2)$.

Main steps in the proof

- The purity theorem for the Brauer group to show that the quaternion algebra $\mathcal{A}$ defines an element in $\mathrm{Br}(X)$.
- Find two elements $x_1, x_2 \in X(\mathbb{Q}_2)$ such that $\mathrm{ev}_{\mathcal{A}}(x_1) \neq \mathrm{ev}_{\mathcal{A}}(x_2)$.
- To show that $X(\mathbb{Q})$ is not dense in $X(\mathbb{Q}_2)$ we prove that for every place ν different from 2 the evaluation map $\mathrm{ev}_{\mathcal{A}} : X(\mathbb{Q}_\nu) \to \mathrm{Br}(\mathbb{Q}_\nu)$ is constant and trivial.

Transcendental nature of $\mathcal{A}$

Let $\mathrm{Br}_1(X)$ be the algebraic Brauer group of X (i.e. $\mathrm{Br}_1(X) = \ker(\mathrm{Br}(X) \to \mathrm{Br}(X \times_k \bar{k}))$. Colliot-Thélène and Skorobogatov have proven that for every element $\mathcal{B} \in \mathrm{Br}_1(X)$ and every finite field extension $k/\mathbb{Q}_p$, where p is a prime with good reduction $\mathrm{ev}_{\mathcal{B}} : X(k) \to \mathrm{Br}(k)$ has to be constant. Hence, $\mathcal{A} \notin \mathrm{Br}_1(X)$ (i.e. $\mathcal{A}$ is a transcendental element of $\mathrm{Br}(X)$).

Construction of the quaternion algebra $\mathcal{A}$

Let Y be the K3 surface over $\mathbb{F}_2$ defined by the reduction modulo 2 of the equation defining X.

1. $\mathrm{H}^0(Y, \Omega^2_Y)$ has just one non-zero element ω

$$\omega = \frac{d\left(\frac{z^3+w^2x+xyz}{x^3}\right)}{\frac{z^3+w^2x+xyz}{x^3}} \wedge \frac{d\left(\frac{z}{x}\right)}{\frac{z}{x}} = \frac{df}{f} \wedge \frac{dg}{g}.$$

2. Crucial step:

$$\mathrm{Br}(k(X)) \ni \mathcal{A} := \left(\frac{z^3+w^2x+xyz}{x^3}, -\frac{z}{x}\right) = (\tilde{f}, \tilde{g})$$

defines an element in $\mathrm{Br}(X)$.

The isomorphism of Bloch and Kato

- Let $F = \mathbb{F}_2(Y)$. Then the subgroup of logarithmic q-forms $\Omega^q_{F,\log} \subseteq \Omega^q_F$ is the subgroup generated by elements of the form

$$\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_q}{y_q}, \text{ where } y_i \in F^\times.$$

- Let X_2 be the base change of X to $\mathbb{Q}_2$ and $K = k(X_2)$ its function field. Bloch and Kato define a decreasing filtration $\{U^m \mathrm{H}^q(K, \mu_p^{\otimes q})\}_{m \in \mathbb{N}}$ on $\mathrm{H}^q(K, \mu_p^{\otimes q})$ such that the following theorem holds true.

Theorem [Bloch–Kato]

There exists an isomorphism

$$\rho_0 : \Omega^q_{F,\log} \oplus \Omega^{q-1}_{F,\log} \xrightarrow{\sim} \frac{\mathrm{H}^q(K, \mu_p^{\otimes q})}{U^1 \mathrm{H}^q(K, \mu_p^{\otimes q})} =: \mathrm{gr}^0$$

Suppose that K contains a primitive p-root of unity, then we get an isomorphism between $\mu_p^{\otimes 2}$ and μ_p. Hence,

$$\rho_0 : \Omega^2_{F,\log} \oplus \Omega^1_{F,\log} \xrightarrow{\sim} \frac{\mathrm{H}^2(K, \mu_p^{\otimes 2})}{U^1\mathrm{H}^2(K, \mu_p^{\otimes 2})} \simeq \frac{\mathrm{Br}(K)[p]}{U^1\,\mathrm{Br}(K)[p]}$$

In particular, it maps $\frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2}$ to the class of the quaternion algebra $(\tilde{y}_1, \tilde{y}_2)$, where $\tilde{y}_i$ is any lift of y_i to K.

Back to the quaternion algebra $\mathcal{A}$

1. $\omega \in \mathrm{H}^0(Y, \Omega^2_Y)$ is of the form

$$\omega = \frac{d\left(\frac{z^3+w^2x+xyz}{x^3}\right)}{\frac{z^3+w^2x+xyz}{x^3}} \wedge \frac{d\left(\frac{z}{x}\right)}{\frac{z}{x}} \in \Omega^2_{F,\log}.$$

2. $\mathbb{Q} \subseteq K$, hence K contains a primitive 2-root of unity. Moreover, we have

$$\rho_0 : \Omega^2_{F,\log} \oplus \Omega^1_{F,\log} \to \frac{\mathrm{Br}(K)[2]}{U^1\,\mathrm{Br}(K)[2]}$$

$$(\omega, 0) \mapsto [\mathcal{A}] = \left[\left(\frac{z^3+w^2x+xyz}{x^3}, -\frac{z}{x}\right)\right].$$

Hence, the image of $\mathcal{A}$ in $\mathrm{Br}(K)[2]$ does not lie in $U^1\,\mathrm{Br}(K)[2]$.

The Evaluation filtration

Bright and Newton define the following filtration on $\mathrm{Br}(X_2)$

$$\mathrm{Ev}_n \mathrm{Br}(X_2) := \{\mathcal{A} \in \mathrm{Br}(X_2) \mid \forall L/\mathbb{Q}_2, \forall P \in X_2(L),$$
$$\mathrm{ev}_{\mathcal{A}} \text{ is constant on } B(P, e_{L/\mathbb{Q}_2} n + 1)\}.$$

Theorem [Bright–Newton 2020]

$$\mathrm{Ev}_0 \mathrm{Br}(X_2)[2] = \{\mathcal{B} \in \mathrm{Br}(X_2)[2] \mid \mathcal{B} \in U^2 \mathrm{Br}(K)[2]\}.$$

Hence $\mathcal{A} \notin U^1 \mathrm{Br}(K)[2]$ implies that $\mathcal{A} \notin \mathrm{Ev}_0 \mathrm{Br}(X_2)[2]$.

Conclusion

The two main ideas in this example were the following:

1. Use Bloch and Kato isomorphism together with the results of Bright and Newton to deduce that if $\mathcal{A} \in \mathrm{Br}(X)[2]$ is such that $\mathrm{ev}_{\mathcal{A}} : X(\mathbb{Q}_2) \to \mathrm{Br}(\mathbb{Q}_2)$ is non-constant, then $\mathcal{A}$ should come from the only non-zero element $\omega \in \mathrm{H}^0(Y, \Omega^2_Y)$.
2. Find an element $\mathcal{A}$ in $\mathrm{Br}(K)[2]$ whose image in $\frac{\mathrm{Br}(K)[2]}{U^1 \mathrm{Br}(K)[2]}$ is $\rho_0(\omega, 0)$ and that defines also an element in $\mathrm{Br}(X)$.

Thank you for the attention!