ZORP

Jordan Ellenzorp
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"questionZ around cOunting Rational Points
(on stacks)"

Q| A long list of questions (and some answers)

Q: How many isomorphism classes of elliptic curves $E/\mathbb{Q}$ are there such that:

- E has a rational 5-isogeny
- $\max(|A|^3, |B|^2) < X$

($E: y^2 = x^3 + Ax + B$ minimal Weierstrass form)

A: $\Theta(X^{1/6} \log^2 X)$

(Boggess-Sankar, 2020)

(Remark: for rational 5-torsion instead of 5-isogeny, $\Theta(X^{1/6})$ by Harron-Snowden (2017))

Also in Boggess-Sankar: N-isogenies for $N = 2, 3, 4, 6, 8, 9, 12, 16, 18$.

Q: What about 7-isogenies?

Define

$$a^{\lceil \frac{1}{m} \rceil} = \prod_p p^{\lceil \frac{1}{m} \mathrm{ord}_p a \rceil}$$

$$a = n^m \implies a^{\lceil \frac{1}{m} \rceil} = a^{1/m}$$

$$a \text{ squarefree} \implies a^{\lceil \frac{1}{m} \rceil} = a$$

Def: $$a^{\lceil \frac{1}{m} \rceil - \frac{1}{m}} = a^{\lceil \frac{1}{m} \rceil} |a|^{-\frac{1}{m}}$$

e.g. $$a^{\lceil \frac{1}{2} \rceil - \frac{1}{2}} = \left(\frac{|a|}{\text{largest square dividing } a} \right)^{1/2} =: \mathrm{sqf}(a)^{\frac{1}{2}}$$

Choose $m_0, m_1, m_\infty \in \mathbb{Z}$

For each a, b coprime, define

$$H_{m_0, m_1, m_\infty}(a,b) = |a|^{\lceil \frac{1}{m_0} \rceil - \frac{1}{m_0}} \, |b|^{\lceil \frac{1}{m_\infty} \rceil - \frac{1}{m_\infty}} \, |a-b|^{\lceil \frac{1}{m_1} \rceil - \frac{1}{m_1}} \, \max(|a|,|b|)^{\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_\infty} - 1}$$

How many pairs (a,b) with

$$H_{m_0, m_1, m_\infty}(a,b) \leq X \, ?$$

e.g. $(m_0 = m_1 = m_\infty = 1)$

$$\max(|a|,|b|)^2 < X \qquad \sim X$$

$(m_0 = 3,\ m_\infty = m_1 = 1)$

$$a^{\lceil \frac{1}{3} \rceil - \frac{1}{3}} \max(|a|,|b|) < X \qquad \sim X \log^2 X$$

$(m_0 = m_1 = m_\infty = 2)$

$$\mathrm{sqf}(a)^{1/2}\,\mathrm{sqf}(b)^{1/2}\,\mathrm{sqf}(a-b)^{1/2} \max(|a|,|b|)^{1/2} < X$$

Le Boudec (2020)

Nasserden-Xiao (2020)

$c_1 X \log^3 X$ < count < $c_2 X \log^3 X$

Q: general m_0, m_1, m_∞?

with $\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_\infty} - 1 > 0$

Q: How many points $P \in \mathbb{P}^2(\overline{\mathbb{Q}})$ such that

- $Ht_{abs}(P) < X$
- $[\mathbb{Q}(P) : \mathbb{Q}] = 3$?

Guignard (2017): $\sim X^{12}$, but

P ~~not~~ on a $\mathbb{Q}$-rational line

$\sim X^{9+\varepsilon}$

Questions you've heard me talk about before:

How many extensions $K/\mathbb{Q}$ with Galois group G and discriminant (or other ramification invariant) $< X$? (Malle's conjecture and variants)

(What about: with an $\alpha \in \mathcal{O}_K^{-2}$ with all archimedean absolute values $< \Delta_K^{.01}$?)

(Harron H, _)

All are questions of the form:

$\mathcal{X}$ a smooth proper Deligne-Mumford stack

$\mathcal{V}$ a vector bundle on $\mathcal{X}$

How many points $P \in \mathcal{X}(\mathbb{Q})$ with height $< X$?

E_, Satriano, Zureick-Brown

(Sounds like Batyrev-Manin...)

(Note: height here is with respect to a vector bundle V on $\mathcal{X}$, not necessarily a line bundle!)

Counting 5-isogenies:

Difficulty: many elliptic curves with the same j-invariant and a 5-isogeny.

$$\mathcal{X} = X_0(5)$$

$\mathcal{V}$ = Hodge bundle

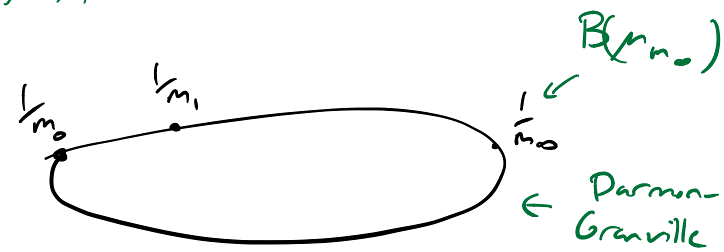
$$\text{height}\,(y^2 = x^3 + Ax + B) = \max\left(|A|^3, |B|^2\right)^{1/12}$$

(soi-disant "naive" height)

Voight
DZB, "stacky curves"

H_{m_0,m_1,m_∞}:

$\mathcal{X} =$



H_{m_0,m_1,m_∞} = anticanonical height

Batyrev-Manin philosophy: "$\sim X^{1+\varepsilon}$ points of anticanonical height $< X$"

Our conjecture: for all m_0, m_1, m_∞ with

$$\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_\infty} - 1 > 0,$$

$\#(a,b)$: $Ht_{m_0,m_1,m_\infty}(a,b)$ is between X and $X^{1+\varepsilon}$

(presumably $\sim C X (\log X)^L$)

Note: $\mathcal{X}$ is birational to $\mathbb{P}^1$, and $\mathcal{X}(\mathbb{Q})$ and $\mathbb{P}^1(\mathbb{Q})$ are not very different!

$$\text{“}g(\mathcal{X})\text{''} = \frac{1}{2}\left(3 - \frac{1}{m_0} - \frac{1}{m_1} - \frac{1}{m_\infty}\right)$$

So our

$$\mathrm{sqf}(a)\,\mathrm{sqf}(b)\,\mathrm{sqf}(b-a)\max(|a|,|b|)$$

example is a curve of genus $3/4$

Bhargava-Poonen (2020): Stacky curves of genus $< \frac{1}{2}$ have a local-to-global principle for integral points. Christensen (2020): strong approx, too!

WARNING:

“$\#\, P \in \mathcal{X}(\mathbb{Q})$: anticanonical height $< X$”

is not always $\sim c X^{1+\varepsilon}$; when $\mathcal{X} = BG$, anticanonical height is 0!

$\mathbb{*}/G$

Cubic points on $\mathbb{P}^2$:

$$\mathcal{X} = (\mathbb{P}^2)^3 / S_3$$

Guignard's $X^{9+\varepsilon}$ agrees with our conjecture

"Cubic points on $\mathbb{P}^2$ contained in a $\mathbb{Q}$-rational line" form a closed (accumulating) locus in $\mathcal{X}$!

(Le Rudulier)

WHY AM I NOT STATING THE CONJECTURE???

Key element of Batyrev-Manin:

Fujita invariant

$$a(L) = \min\{t \in \mathbb{R} : K_X + tL \text{ effective}\}$$

What does "effective" mean for a vector bundle on a stack??

Our current hacky approach: roughly,

"V is effective if $h_V(P)$ is bounded below on a dense open of $\mathcal{X}$, as P ranges over algebraic points of degree $\leq d$"

(Northcott property)

THERE SHOULD BE A BETTER WAY.

Gives: Malle conjecture and many known variants

($\mathcal{X} = BG$) ← Yasuda

usual Batyrev-Manin

examples in this talk

Not discussed:

relation with Abramovich-Várilly

relation with Peyre and "freeness"

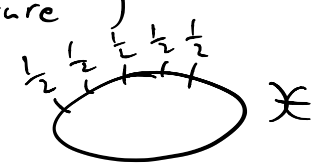
analogues of Vojta's conjecture

$\boxed{\text{e.g.}}$ are there only finitely many primitive 5-term APs

with

$\mathrm{sqf}(a_1 a_2 a_3 a_4 a_5) < \max |a_i|^{1/2 - \delta}$?

(cf. Vojta's "more general abc conjecture")



$\boxed{\text{e.g.}}$ control of points in $A_d(\mathbb{Q})$

(quadratic twists of abelian 3-fold) with height

$< d^c$?

$A/\pm 1$

Malle predicts:

$\#$ G-extns, disc < X
of K

$$\sim C_{K,G} X^{a(G)} (\log X)^{b(G,K)}$$

Which vector bundle on BG?

(i.e. which rep'n of G)

Regular rep of G $\longrightarrow$ disc. of Galois G-extension

Perm rep of S_n $\longrightarrow$ disc of degree-n extension with Galois closure having group S_n.

2-dim'l rep of D_4 $\longrightarrow$ new count of D_4-extns by Varma, Altug, Shankar, Wilson