On the distribution of Campana points on toric varieties (joint work with Marta Pieropan)

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Introduction

Motivation

Interpolate between integral and rational points on varieties over number fields.

Definition

Let $m \geq 1$. We call an integer $a \in \mathbb{Z}$ m-full if

$$p \mid a \Rightarrow p^m \mid a, \quad p \text{ prime}$$

Example

Let $[x_0 : x_1]$ be homogeneous coordinates for $\mathbb{P}^1_{\mathbb{Z}}$ and $\mathcal{D} = \{x_1 = 0\}$.

- Integral points on $\mathbb{P}^1_{\mathbb{Z}} \setminus \mathcal{D}$ correspond to $x_0 \in \mathbb{Z}$ and $x_1 \in \{\pm 1\}$
- Campana points $(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{D}, m)$ correspond to $x_0 \in \mathbb{Z}$ and $x_1 \in \mathbb{Z}$ m-full, $\gcd(x_0, x_1) = 1$.

Campana points

Definition

Let X be a smooth proper variety over a number field k and $D = \cup_{i=1}^n D_i$ a strict normal crossing divisor. Let $S \subset \Omega_k$ be a finite set of places of k such that there exists a smooth proper $\mathcal{O}_{k,S}$-model $(\mathcal{X}, \mathcal{D})$ of (X, D) and such that $\mathcal{D} = \cup_{i=1}^n \mathcal{D}_i$ is a strict normal crossing divisor modulo primes p not contained in S. Let $\mathbf{m} = (m_1, \ldots, m_n) \in (\mathbb{Z}_{\geq 0})^n$. Define

$$(\mathcal{X}, \mathcal{D}, \mathbf{m})(\mathcal{O}_{k,S}) = \{x \in \mathcal{X}(\mathcal{O}_{k,S}), x \notin D, \\ \forall_{p \notin S} \nu_p(x^* \mathcal{D}_i) > 0 \Rightarrow \nu_p(x^* \mathcal{D}_i) \geq m_i,\ 1 \leq i \leq n\}.$$

Campana points

Definition

Let X be a smooth proper variety over a number field k and $D = \cup_{i=1}^n D_i$ a simple normal crossing divisor. Let $S \subset \Omega_k$ be a finite set of places of k such that there exists a smooth proper $\mathcal{O}_{k,S}$-model $(\mathcal{X}, \mathcal{D})$ of (X, D) and such that $\mathcal{D} = \cup_{i=1}^n \mathcal{D}_i$ is a simple normal crossing divisor modulo primes p not contained in S. Let $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0}$. Define

$$(\mathcal{X}, \mathcal{D}, \mathbf{m})(\mathcal{O}_{k,S}) = \{x \in \mathcal{X}(\mathcal{O}_{k,S}), x \notin D, \\ \forall_{p \notin S} \nu_p(x^*\mathcal{D}_i) > 0 \Rightarrow \nu_p(x^*\mathcal{D}_i) \geq m_i,\ 1 \leq i \leq n\}.$$

Remark

If f_i is a local equation of $\mathcal{D}_i$ around x then

$$\nu_p(x^*\mathcal{D}_i) = \nu_p(f_i(x)).$$

Campana points, an example

Example

Let $m \geq 1$. Let $[x_0 : x_1]$ be homogeneous coordinates for $\mathbb{P}^1_{\mathbb{Z}}$ and $\mathcal{D} = \{x_1 = 0\}$. Then

$$(\mathbb{P}^1_{\mathbb{Z}}, \{x_1 = 0\}, m)(\mathbb{Z}) = \{(x_0 : x_1), x_0, x_1 \in \mathbb{Z} \text{ coprime }, x_1 \text{ is } m\text{−full}\}.$$

One has the inclusions

$$(\mathbb{P}^1_{\mathbb{Z}} \setminus \mathcal{D})(\mathbb{Z}) \subset (\mathbb{P}^1_{\mathbb{Z}}, \mathcal{D}, m)(\mathbb{Z}) \subset (\mathbb{P}^1_{\mathbb{Q}} \setminus D)(\mathbb{Q}).$$

Counting Campana points?

Question

Assume that $(\mathcal{X}, \mathcal{D}, \mathbf{m})(\mathcal{O}_{k,S})$ is Zariski-dense in X (and not thin). What can we say about the distribution of the points $(\mathcal{X}, \mathcal{D}, \mathbf{m})(\mathcal{O}_{k,S})$ in X? Assume we are given a suitable height function, what should one expect for the number of Campana points up to a certain height?

Counting Campana points

Theorem (Van Valckenborgh 2012)

Take $k = \mathbb{Q}$, $X = \mathbb{P}^{n-1}$ *and* $\Delta = H_0 \cup \ldots \cup H_n$, *with*

$$H_i = \{x_i = 0\}, \quad 0 \leq i \leq n-1,$$

and

$$H_n = \{x_0 + \ldots + x_{n-1} = 0\}.$$

Set $m_0 = \ldots = m_n = 2$. *Then points in* $(\mathcal{X}, \mathcal{D}, \mathbf{m})(\mathbb{Z})$ *correspond to tuples*

$$(x_0, \ldots, x_{n-1}) \in \mathbb{Z}^n, \quad \gcd(x_0, \ldots, x_{n-1}) = 1$$

such that x_i *is square-full for* $0 \leq i \leq n-1$ *and*

$$x_0 + \ldots + x_{n-1} \text{ is squarefull.}$$

Counting Campana points

Theorem (Van Valckenborgh 2012)

Define the height function

$$H(x_0 : \ldots : x_{n-1}) := \max\{|x_0|, \ldots, |x_{n-1}|, |\sum_{i=0}^{n-1} x_i|\},$$

on coprime tuples $(x_0, \ldots x_{n-1}) \in \mathbb{Z}^n$*. Then for* $n \geq 4$ *one has*

$$\sharp\{x \in (\mathcal{X}, \Delta, \mathbf{m})(\mathbb{Z}) : H(x) \leq B\} = CB^{\frac{n-1}{2}} + O\left(B^{\frac{n-1}{2}-\delta}\right).$$

Basically, one needs to count square-full solutions to

$$x_0 + \ldots + x_{n-1} = x_n.$$

m-full numbers

Theorem (Erdos-Szekeres 1935)

Let $m \geq 1$. Then

$$\sharp\{1 \leq y \leq B : y \text{ is } m - \text{full}\} \sim c_m B^{\frac{1}{m}}$$

Idea: parametrize m-full numbers by products $\prod_{r=0}^{m-1} y_r^{m+r}$ with $y_1, \ldots, y_{m-1}$ square-free and pairwise coprime.

Lemma (Pieropan-S 2020)

Let $d > 0$ be square-free, $m \geq 2$. Then

$$\sharp\{1 \leq y \leq B : y \text{ is } m - \text{full}, \ d \mid y\}$$
$$\sim c_m B^{\frac{1}{m}} \prod_{p \mid d} \left(1 + p - p^{\frac{m-1}{m}}\right)^{-1}.$$

Counting Campana points

Let $X = \mathbb{P}^{n-1}$, $\Delta = \cup_{i=0}^{n} D_i$ with $D_i = \{x_i = 0\}$, $0 \leq i \leq n-1$, and $D_n = \{c_0 x_0 + \ldots + c_{n-1} x_{n-1} = 0\}$, for $c_0, \ldots, c_{n-1} \in \mathbb{Z} \setminus \{0\}$.

Theorem (Browning-Yamagishi 2019)

Assume that $m_0, \ldots, m_n \geq 2$ such that there exists $j \in \{0, \ldots, n\}$ with

$$\sum_{\substack{0 \leq i \leq n \\ i \neq j}} \frac{1}{m_i(m_i+1)} \geq 1.$$

Then

$$\sharp\{x \in (\mathcal{X}, \Delta, \mathbf{m})(\mathbb{Z}) : H_{naiv}(x) \leq B\} \sim c B^{\sum_{i=0}^{n} \frac{1}{m_i} - 1}.$$

Note

$$K_{\mathbb{P}^{n-1}} + \sum_{i=0}^{n} \left(1 - \frac{1}{m_i}\right) D_i \sim \left(1 - \sum_{i=0}^{n} \frac{1}{m_i}\right) H.$$

Counting Campana points

Question

Conjectures for the growth of the number of Campana points of bounded height?

Conjecture (Manin-Peyre)

Let V be a smooth projective Fano variety over a number field k such that $V(k)$ is dense in V. Then there exists a thin subset Z such that

$$\sharp\{x \in V(k) \setminus Z : H_{\omega_X^{-1}}(x) \leq B\} \sim cB(\log B)^{\mathrm{rk}(\mathrm{Pic}(X))-1}.$$

Manin-type conjecture for Campana points

Assume that $(\mathcal{X}, \mathcal{D}, \mathbf{m})(\mathcal{O}_{k,S})$ is Zariski-dense in X (and not thin), and let L be an ample line bundle on X.

Conjecture (Pieropan-Smeets-Tanimoto-Varilly-Alvarado 2019)

There exists a thin set Z such that

$$\sharp\{x \in (\mathcal{X}, \mathcal{D}, \mathbf{m})(\mathcal{O}_{k,S}) \setminus Z : H_L(x) \leq B\} \sim cB^a (\log B)^{b-1},$$

where c is a product of local densities,

$$a = \inf\{t \in \mathbb{R} : tL + K_X + \sum_{i=1}^{n} \left(1 - \frac{1}{m_i}\right) D_i \textit{ is effective } \},$$

and b is the codimension of the minimal face of the effective cone that contains $aL + K_X + \sum_{i=1}^{n} \left(1 - \frac{1}{m_i}\right) D_i$.

Campana points on toric varieties

Theorem (Pieropan-S. 2020)

Let X be a split smooth proper toric variety over $\mathbb{Q}$ with boundary divisor $D = \cup_{i=1}^s D_i$. Let $m_i \geq 2$ for $1 \leq i \leq s$ and assume that $L = -\left(K_X + \sum_{i=1}^s \left(1 - \frac{1}{m_i}\right) D_i\right)$ is ample + a technical condition on L. Let $r = \operatorname{rank} \mathsf{Pic}(X)$. Then

$$\sharp\{x \in (\mathcal{X}, \mathcal{D}, \mathbf{m})(\mathbb{Z}) : H_L(x) \leq B\} \sim cB(\log B)^{r-1}.$$

where c is compatible with the conjectured constant.

Remark

The technical condition holds for e.g. projective space, products of projective spaces, blow-up of $\mathbb{P}^2$ in one point, and all smooth projective toric varieties with $\operatorname{rank} \mathsf{Pic}(X) \geq \dim X + 2$.

Proof strategy:

- Use Cox rings/universal torsor method
- Generalized version of the Blomer-Brüdern hyperbola method

Future goals: allow for the removal of more general divisors, consider hypersurfaces within toric varieties

Let $Y \to X$ be the universal torsor of X. Then $Y \subset \mathbb{A}^s_{\mathbb{Q}} = \mathrm{Spec}(\mathbb{Q}[y_{\rho_1}, \ldots, y_{\rho_s}])$ is the open subvariety given by the complement of

$$\langle \prod_{\rho \notin \sigma} y_\rho = 0,\ \sigma \in \Sigma_{\mathsf{max}} \rangle.$$

Let $\pi : \mathcal{Y} \to \mathcal{X}$ be an integral model of $Y \to X$. By Salberger's work

$$\pi^{-1}((\mathcal{X}, \mathcal{D}, \mathbf{m}))(\mathbb{Z}) = \{\mathbf{y} \in \mathcal{Y}(\mathbb{Z}) : y_i \neq 0,\ y_i \text{ if } m_i\text{−full},\ 1 \leq i \leq s\}.$$

Remark

Finding $\mathbf{y} \in \mathcal{Y}(\mathbb{Z})$ translates into finding tuples $(y_1, \ldots, y_s) \in \mathbb{Z}^s$ with

$$\gcd\left(\prod_{\rho \notin \sigma} y_\rho, \sigma \in \Sigma_{\max}\right) = 1.$$

The height function

Let $L = -\left(K_X + \sum_{i=1}^{s}\left(1 - \frac{1}{m_i}\right) D_i\right)$, and assume that L is (very) ample.
For $\sigma \in \Sigma_{\max}$ find a divisor $L(\sigma) \sim L$ with

$$L(\sigma) = \sum_{\rho_i \notin \sigma} \alpha_{\sigma,i} D_i.$$

Proposition

Let $\mathbf{y} \in Y(K)$. Then

$$H_L(\pi(\mathbf{y})) = \prod_{\nu \in \Omega_k} \sup_{\sigma \in \Sigma_{\max}} \prod_{i=1}^{s} |y_i^{\alpha_{\sigma,i}}|_\nu.$$

The counting function

Goal

Asymptotically evaluate the counting function

$$N(B) = \frac{1}{2^r} \sharp\{\mathbf{y} \in \mathcal{Y}(\mathbb{Z}) : y_i \neq 0,\ \ y_i \text{ is } m_i - full, 1 \leq i \leq s,$$
$$\max_{\sigma \in \Sigma_{\max}} \prod_{i=1}^{s} |y_i|^{\alpha_{\sigma,i}} \leq B\}.$$

An example

Let $X = \mathbb{P}^1 \times \mathbb{P}^1$.
The universal torsor $Y \subset \mathbb{A}^4 = \mathrm{Spec}(\mathbb{Q}[x_0, y_0, x_1, y_1])$ is given by the complement of the subvariety given by $\langle x_0 y_0, y_0 x_1, x_1 y_1, y_1 x_0 \rangle$, i.e.

$$Y = \mathbb{A}^4 \setminus (\{x_0 = x_1 = 0\} \cup \{y_0 = y_1 = 0\}) .$$

Take $m_1 = \ldots = m_4 = 2$ and $K_x = -\sum_{i=1}^4 D_i$, i.e. $L = \frac{1}{2} \sum_{i=1}^4 D_i$. Then the height function for integral points $(x_0, y_0, x_1, y_1) \in \mathcal{Y}(\mathbb{Z})$ is given by

$$\begin{aligned} H_L(\pi(x_0, y_0, x_1, y_1)) &= \max(|y_0 x_0|, |y_0 x_1|, |y_1 x_0|, |y_1 x_1|) \\ &= \max(|x_0|, |x_1|) \max(|y_0|, |y_1|). \end{aligned}$$

Expectation for the growth of $N(B)$

$$N(B) = \frac{1}{2^r}\sharp\{\mathbf{y} \in \mathcal{Y}(\mathbb{Z}) : y_i \neq 0,\ y_i \text{ is } m_i - \text{full}, 1 \leq i \leq s,$$

$$\max_{\sigma \in \Sigma_{\max}} \prod_{i=1}^{s} |y_i|^{\alpha_{\sigma,i}} \leq B\}.$$

Idea: consider the contribution of a dyadic box

$$B_i \leq y_i < 2B_i, \quad 1 \leq i \leq s.$$

Let $B_i = B^{t_i}$ for $t_i \geq 0$. Then

$$\sharp\{(y_1, \ldots, y_s) \in \mathbb{Z}^2 : y_i \sim B_i, m_i - \text{full},\ 1 \leq i \leq s\}$$

$$\sim C \prod_{i=1}^{s} B_i^{\frac{1}{m_i}} \sim C B^{\sum_{i=1}^{s} \frac{1}{m_i} t_i}.$$

Expectation for the growth of $N(B)$

Idea: consider the contribution of a dyadic box

$$B_i \leq y_i < 2B_i, \quad 1 \leq i \leq s.$$

For the height condition to hold

$$\max_{\sigma \in \Sigma_{\max}} \prod_{i=1}^{s} |y_i|^{\alpha_{\sigma,i}} \leq B$$

we consider boxes for which

$$\prod_{i=1}^{s} B_i^{\alpha_{\sigma,i}} \leq B, \quad \forall \sigma \in \Sigma_{\max}.$$

I.e. we consider $B_i = B^{t_i}$, with

$$\sum_{i=1}^{s} \alpha_{\sigma,i} t_i \leq 1, \quad \sigma \in \Sigma_{\max},$$

$$t_i \geq 0, \quad 1 \leq i \leq s.$$

Maximizing a linear function on a polytope

Let $\mathcal{P} \subset \mathbb{R}^s$ be the polytope given by

$$\begin{aligned} \sum_{i=1}^{s} \alpha_{\sigma,i} t_i \leq 1, &\quad \sigma \in \Sigma_{\max}, \\ t_i \geq 0, &\quad 1 \leq i \leq s. \end{aligned}$$

Goal

Maximize the function $\sum_{i=1}^{s} \frac{1}{m_i} t_i$ on the polytope $\mathcal{P}$.

- linear programming problem

Remark

Expected log exponent = dimension of the face of the polytope $\mathcal{P}$ where the max is attained.

The conjectured exponent

$$a = \inf\{t \in \mathbb{R} : tL + K_X + \sum_{i=1}^{s} \left(1 - \frac{1}{m_i}\right) \text{ is effective}\},$$

leads to the following linear programming problem.

Minimize the linear function $\sum_{\sigma \in \Sigma_{\max}} \lambda_\sigma$ *subject to the conditions*

$$\lambda_\sigma \geq 0, \quad \sigma \in \Sigma_{\max}$$

$$\sum_{\sigma \in \Sigma_{\max}} \lambda_\sigma \alpha_{i,\sigma} \geq \frac{1}{m_i}, \quad 1 \leq i \leq s.$$

Duality in linear programming

Theorem (Strong duality in linear programming)

Let $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$, $\mathbf{b} \in \mathbb{R}^m$ *and* $\mathbf{c} \in \mathbb{R}^n$*.*
$\mathcal{P}$*: Maximize* $\mathbf{c}^t\mathbf{x}$ *subject to*

$$A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq 0.$$

$\mathcal{D}$*: Minimize* $\mathbf{b}^t\mathbf{y}$ *subject to*

$$A^t\mathbf{y} \geq \mathbf{c}, \quad \mathbf{y} \geq 0.$$

If $\mathcal{P}$ *has a finite optimal solution then so does* $\mathcal{D}$ *and these two are equal.*

A pair of dual linear programming problems

The exponent that we compute

Maximize the function $\sum_{i=1}^{s} \frac{1}{m_i} t_i$ *subject to*

$$\sum_{i=1}^{s} \alpha_{\sigma,i} t_i \leq 1, \quad \sigma \in \Sigma_{\max},$$

$$t_i \geq 0, \quad 1 \leq i \leq s.$$

The conjectured exponent

Minimize the linear function $\sum_{\sigma \in \Sigma_{\max}} \lambda_\sigma$ *subject to the conditions*

$$\lambda_\sigma \geq 0, \quad \sigma \in \Sigma_{\max}$$

$$\sum_{\sigma \in \Sigma_{\max}} \lambda_\sigma \alpha_{i,\sigma} \geq \frac{1}{m_i}, \quad 1 \leq i \leq s.$$

From box counting to hyperbola shapes

Let $f : \mathbb{N}^s \to \mathbb{R}_{\geq 0}$ be an arithmetic function. Assume that we understand sums of f over boxes. Let B be a large real parameter, $\mathcal{K}$ a finite index set and $\alpha_{i,k} \geq 0$ for $1 \leq i \leq s$ and $k \in \mathcal{K}$.

Goal

Find an asymptotic for

$$S^f := \sum_{\substack{\prod_{i=1}^s y_i^{\alpha_{i,k}} \leq B, \ \forall k \in \mathcal{K} \\ y_i \in \mathbb{N}, 1 \leq i \leq s}} f(\mathbf{y}).$$

Remark

We don't assume any multiplicative structure for f.

From box counting to hyperbola shapes

Property I

Assume that there are non-negative real constants $C_{f,M} \leq C_{f,E}$ *and* $\delta > 0$ *and* $\varpi_i > 0$, $1 \leq i \leq s$ *such that for all* $B_1, \ldots, B_s \in \mathbb{R}_{\geq 1}$ *we have*

$$\sum_{\substack{1 \leq y_i \leq B_i \\ 1 \leq i \leq s}} f(\mathbf{y}) = C_{f,M} \prod_{i=1}^{s} B_i^{\varpi_i} + O\left(C_{f,E} \prod_{i=1}^{s} B_i^{\varpi_i} \left(\min_{1 \leq i \leq s} B_i \right)^{-\delta} \right)$$

where the implied constant is independent of f.

From box counting to hyperbola shapes

Property II

Assume that there are positive real numbers D and ν such that the following holds. Let $\mathcal{I} \subsetneq \{1, \ldots, s\}$ be a non-empty subset of indices and fix some $(y_i)_{i \in \mathcal{I}} \in \mathbb{N}^{|\mathcal{I}|}$. Write $\mathbf{y}_\mathcal{I}$ for the vector $(y_i)_{i \in \mathcal{I}}$ and $|\mathbf{y}_\mathcal{I}|$ for its maximums norm. Then there is a non-negative constant $C_{f,M,\mathcal{I}}(\mathbf{y}_\mathcal{I})$ such that for all $B_i \in \mathbb{R}_{\geq 1}$, $i \in \{1, \ldots, s\} \setminus \mathcal{I}$ one has

$$\sum_{1 \leq y_i \leq B_i,\ i \notin \mathcal{I}} f(\mathbf{y}) = C_{f,M,\mathcal{I}}(\mathbf{y}_\mathcal{I}) \prod_{i \notin \mathcal{I}} B_i^{\varpi_i} + O(C_{f,E} |\mathbf{y}_\mathcal{I}|^D \prod_{i \notin \mathcal{I}} B_i^{\varpi_i} (\min_{i \notin \mathcal{I}} B_i)^{-\delta}),$$

uniformly in $|\mathbf{y}_\mathcal{I}| \leq (\prod_{i \notin \mathcal{I}} B_i)^\nu$.

Recall

$$S^f := \sum_{\substack{\prod_{i=1}^s y_i^{\alpha_{i,k}} \leq B,\ \forall k \in \mathcal{K} \\ y_i \in \mathbb{N}, 1 \leq i \leq s}} f(\mathbf{y}).$$

Define the polyhedron $\mathcal{P} \subset \mathbb{R}^s$ by

$$\sum_{i=1}^{s} \alpha_{i,k} \varpi_i^{-1} t_i \leq 1, \quad k \in \mathcal{K} \tag{0.1}$$

and

$$t_i \geq 0, \quad 1 \leq i \leq s. \tag{0.2}$$

The linear function $\sum_{i=1}^s t_i$ takes its maximal value on a face of $\mathcal{P}$ which we call F. Write a for its maximal value.

From box counting to hyperbola shapes

Theorem (Pieropan-S. 2020)

Let $f : \mathbb{N}^s \to \mathbb{R}_{\geq 0}$ satisfy Property I and Property II. Assume that $\mathcal{P}$ is bounded and non-degenerate, that F is not contained in a coordinate hyperplane of $\mathbb{R}^s$ + a technical condition on $\mathcal{P}$. Let $k = \dim F$. Then we have*

$$S^f = (s-1-k)!\,C_{f,M}c_P(\log B)^k B^a + O\left(C_{f,E}(\log\log B)^s(\log B)^{k-1}B^a\right).$$

Remark

The case $|\mathcal{K}| = 1$, $\alpha_{i,k} = \alpha > 0$ for all $1 \leq i \leq s$ and $k = s-1$ is contained in the original work of Blomer and Brüdern on the hyperbola method.

On the proof

In the hyperbola method, Blomer and Brüdern use the combinatorial identity

$$(1-t)^s \sum_{\substack{j_1+\ldots+j_s\leq J\\ j_i\geq 0}} t^{j_1+\ldots+j_s} = 1 - t^{J+1}\sum_{l=0}^{s-1}\binom{J+l}{l}(1-t)^l,$$

for $t\in\mathbb{C}$ and $J\in\mathbb{N}$.

Problem

Replace the summation condition $j_1+\ldots+j_s\leq J$ by the intersection of a lattice with a polytope.

Idea

Use lattice point counting arguments instead.

Questions?

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