Sums of four squareful numbers

Alec Shute

Browning group, IST Austria

May 14, 2021

Squareful numbers

Definition

A nonzero integer n is *squareful* if $p|n \implies p^2|n$ (example: $72 = 2^3 \times 3^2$).

- Every nonzero squareful number n can be written uniquely in the form $n = x^2y^3$, for $y \in \mathbb{Z}_{\neq 0}$ square-free and $x \in \mathbb{N}$. Therefore

$$\#\{n \leq B : n \text{ squareful}\}$$

$$\asymp \sum_{\substack{|y| \leq B^{1/3} \\ y \text{ square-free}}} \#\{x \in \mathbb{N} : x^2y^3 \leq B\} \asymp \sum_{\substack{|y| \leq B^{1/3} \\ y \text{ square-free}}} \left(\frac{B^{1/2}}{|y|^{3/2}} + O(1) \right) \asymp B^{1/2}.$$

Squareful numbers

Definition

A nonzero integer n is *squareful* if $p|n \implies p^2|n$ (example: $72 = 2^3 \times 3^2$).

- Every nonzero squareful number n can be written uniquely in the form $n = x^2y^3$, for $y \in \mathbb{Z}_{\neq 0}$ square-free and $x \in \mathbb{N}$. Therefore

$$\#\{n \leq B : n \text{ squareful}\}$$
$$\asymp \sum_{\substack{|y| \leq B^{1/3} \\ y \text{ square-free}}} \#\{x \in \mathbb{N} : x^2y^3 \leq B\} \asymp \sum_{\substack{|y| \leq B^{1/3} \\ y \text{ square-free}}} \left(\frac{B^{1/2}}{|y|^{3/2}} + O(1)\right) \asymp B^{1/2}.$$

- Define

$$N_k(B) = \#\left\{z_1, \dots, z_k \in \mathbb{Z}_{\neq 0} : \begin{array}{l} z_1, \dots, z_k \text{ squareful}, |z_1|, \dots, |z_k| \leq B, \\ \gcd(z_1, \dots, z_k) = 1, z_1 + \dots + z_k = 0 \end{array}\right\}.$$

Naive heuristic: $N_k(B) \asymp B^{\frac{k}{2}-1}$.

Campana points

- Campana points provide a way to interpolate between rational and integral points.

Campana points

- Campana points provide a way to interpolate between rational and integral points.
- Let K be a field. A *Campana orbifold* is a pair (X, D), where X is a smooth variety over K and
$$D = \sum_{\alpha \in \mathcal{A}} \epsilon_\alpha D_\alpha$$
is an effective Weil $\mathbb{Q}$-divisor of X over K such that
    1. For all $\alpha \in \mathcal{A}$, either $\epsilon_\alpha = 1$ or ϵ_α takes the form $1 - 1/m_\alpha$ for some $m_\alpha \in \mathbb{Z}_{\geq 2}$.
    2. The support $D_{\mathrm{red}} = \sum_{\alpha \in \mathcal{A}} D_\alpha$ of D has strict normal crossings on X.

Campana points

- Campana points provide a way to interpolate between rational and integral points.
- Let K be a field. A *Campana orbifold* is a pair (X, D), where X is a smooth variety over K and
$$D = \sum_{\alpha \in \mathcal{A}} \epsilon_\alpha D_\alpha$$
is an effective Weil $\mathbb{Q}$-divisor of X over K such that
    1. For all $\alpha \in \mathcal{A}$, either $\epsilon_\alpha = 1$ or ϵ_α takes the form $1 - 1/m_\alpha$ for some $m_\alpha \in \mathbb{Z}_{\geq 2}$.
    2. The support $D_{\mathrm{red}} = \sum_{\alpha \in \mathcal{A}} D_\alpha$ of D has strict normal crossings on X.
- Fix a good integral model $(\mathcal{X}, \mathcal{D})$ over $\mathcal{O}_{K,S}$. The Campana points $(\mathcal{X}, \mathcal{D})(\mathcal{O}_{K,S})$ are K-rational points $P \in X(K)$ such that for all α and all places $v \notin S$, the intersection multiplicity of P and $\mathcal{D}_\alpha$ at v is either 0 or $\geq m_\alpha$. (The *intersection multiplicity* of P and $\mathcal{D}_\alpha$ at v is the colength of the pullback of $\mathcal{D}_\alpha$ via $\mathcal{P}_v$ as an ideal in $\mathcal{O}_v$.)

Campana points

- Campana points provide a way to interpolate between rational and integral points.
- Let K be a field. A *Campana orbifold* is a pair (X, D), where X is a smooth variety over K and
$$D = \sum_{\alpha \in \mathcal{A}} \epsilon_\alpha D_\alpha$$
is an effective Weil $\mathbb{Q}$-divisor of X over K such that
  1. For all $\alpha \in \mathcal{A}$, either $\epsilon_\alpha = 1$ or ϵ_α takes the form $1 - 1/m_\alpha$ for some $m_\alpha \in \mathbb{Z}_{\geq 2}$.
  2. The support $D_{\mathrm{red}} = \sum_{\alpha \in \mathcal{A}} D_\alpha$ of D has strict normal crossings on X.
- Fix a good integral model $(\mathcal{X}, \mathcal{D})$ over $\mathcal{O}_{K,S}$. The Campana points $(\mathcal{X}, \mathcal{D})(\mathcal{O}_{K,S})$ are K-rational points $P \in X(K)$ such that for all α and all places $v \notin S$, the intersection multiplicity of P and $\mathcal{D}_\alpha$ at v is either 0 or $\geq m_\alpha$. (The *intersection multiplicity* of P and $\mathcal{D}_\alpha$ at v is the colength of the pullback of $\mathcal{D}_\alpha$ via $\mathcal{P}_v$ as an ideal in $\mathcal{O}_v$.)
- We have $\mathcal{X}^\circ(\mathcal{O}_{K,S}) \subseteq (\mathcal{X}, \mathcal{D})(\mathcal{O}_{K,S}) \subseteq X(K)$.

A Manin-type conjecture for Campana points

- For an irreducible variety X over K, a subset $A \subset X(K)$ is *type I* if A is a proper closed subvariety of X, and *type II* if $A = \varphi(V(K))$, where V is an integral projective variety with $\dim(V) = \dim(X)$ and $\varphi\colon V \to X$ is a dominant morphism of degree at least 2. A *thin* set of $X(K)$ is a subset of a finite union of type I and type II sets. A *thin set of Campana $\mathcal{O}_{K,S}$-points* is the intersection of a thin set of $X(K)$ with the set of Campana points $(\mathcal{X}, \mathcal{D})(\mathcal{O}_{K,S})$.

A Manin-type conjecture for Campana points

- For an irreducible variety X over K, a subset $A \subset X(K)$ is *type I* if A is a proper closed subvariety of X, and *type II* if $A = \varphi(V(K))$, where V is an integral projective variety with $\dim(V) = \dim(X)$ and $\varphi\colon V \to X$ is a dominant morphism of degree at least 2. A *thin* set of $X(K)$ is a subset of a finite union of type I and type II sets. A *thin set of Campana $\mathcal{O}_{K,S}$-points* is the intersection of a thin set of $X(K)$ with the set of Campana points $(\mathcal{X}, \mathcal{D})(\mathcal{O}_{K,S})$.

PSTV-A Conjecture (Pieropan, Smeets, Tanimoto, Várilly-Alvarado)

If (X, D) is a Fano Campana orbifold, and H is a suitable height function, then there is a thin set $\mathcal{T}$ such that

$$\#\{P \in (\mathcal{X}, \mathcal{D})(\mathcal{O}_{K,S}) \backslash \mathcal{T} : H(P) \leq B\} \sim cB^a (\log B)^{b-1}.$$

Example: $N_k(B)$

- Let (X, D) be a Campana orbifold, so $D = \sum_{\alpha \in \mathcal{A}} (1 - 1/m_\alpha) D_\alpha$ for $m_\alpha \in \mathbb{Z}_{\geq 2}$. Fix a good integral model $(\mathcal{X}, \mathcal{D})$ over $\mathcal{O}_{K,S}$. The Campana points $(\mathcal{X}, \mathcal{D})(\mathcal{O}_{K,S})$ are K-rational points $P \in X(K)$ such that for all α and all places $v \notin S$, the intersection multiplicity of P and $\mathcal{D}_\alpha$ at v is either 0 or $\geq m_\alpha$.
- Take the Campana orbifold (X, D), where $X = \mathbb{P}_{\mathbb{Q}}^{k-2}$ and $D = \sum_{i=1}^{k} \frac{1}{2} D_i$, with

$$D_i = \begin{cases} \{z_i = 0\}, & \text{if } 1 \leq i \leq k-1 \\ \{z_1 + \cdots + z_{k-1} = 0\}, & \text{if } i = k. \end{cases}$$

  Choose a height $H(z) = \max(|z_1|, \ldots, |z_{k-1}|, |z_1 + \cdots + z_{k-1}|)$, for a representative $(z_1, \ldots, z_{k-1}) \in \mathbb{Z}_{\text{prim}}^{k-1}$ of $z \in \mathbb{P}^{k-2}(\mathbb{Q})$.

Example: $N_k(B)$

- Let (X, D) be a Campana orbifold, so $D = \sum_{\alpha \in \mathcal{A}} (1 - 1/m_\alpha) D_\alpha$ for $m_\alpha \in \mathbb{Z}_{\geq 2}$. Fix a good integral model $(\mathcal{X}, \mathcal{D})$ over $\mathcal{O}_{K,S}$. The Campana points $(\mathcal{X}, \mathcal{D})(\mathcal{O}_{K,S})$ are K-rational points $P \in X(K)$ such that for all α and all places $v \notin S$, the intersection multiplicity of P and $\mathcal{D}_\alpha$ at v is either 0 or $\geq m_\alpha$.
- Take the Campana orbifold (X, D), where $X = \mathbb{P}_{\mathbb{Q}}^{k-2}$ and $D = \sum_{i=1}^{k} \frac{1}{2} D_i$, with

$$D_i = \begin{cases} \{z_i = 0\}, & \text{if } 1 \leq i \leq k-1 \\ \{z_1 + \cdots + z_{k-1} = 0\}, & \text{if } i = k. \end{cases}$$

  Choose a height $H(z) = \max(|z_1|, \ldots, |z_{k-1}|, |z_1 + \cdots + z_{k-1}|)$, for a representative $(z_1, \ldots, z_{k-1}) \in \mathbb{Z}_{\text{prim}}^{k-1}$ of $z \in \mathbb{P}^{k-2}(\mathbb{Q})$.
- The intersection multiplicities of z and $\mathcal{D}_1, \ldots, \mathcal{D}_k$ at p are respectively the p-adic valuations of $z_1, \ldots, z_{k-1}, z_1 + \cdots + z_{k-1}$.

Example: $N_k(B)$

- Let (X, D) be a Campana orbifold, so $D = \sum_{\alpha \in \mathcal{A}} (1 - 1/m_\alpha) D_\alpha$ for $m_\alpha \in \mathbb{Z}_{\geq 2}$. Fix a good integral model $(\mathcal{X}, \mathcal{D})$ over $\mathcal{O}_{K,S}$. The Campana points $(\mathcal{X}, \mathcal{D})(\mathcal{O}_{K,S})$ are K-rational points $P \in X(K)$ such that for all α and all places $v \notin S$, the intersection multiplicity of P and $\mathcal{D}_\alpha$ at v is either 0 or $\geq m_\alpha$.
- Take the Campana orbifold (X, D), where $X = \mathbb{P}^{k-2}_{\mathbb{Q}}$ and $D = \sum_{i=1}^{k} \frac{1}{2} D_i$, with

$$D_i = \begin{cases} \{z_i = 0\}, & \text{if } 1 \leq i \leq k-1 \\ \{z_1 + \cdots + z_{k-1} = 0\}, & \text{if } i = k. \end{cases}$$

  Choose a height $H(z) = \max(|z_1|, \ldots, |z_{k-1}|, |z_1 + \cdots + z_{k-1}|)$, for a representative $(z_1, \ldots, z_{k-1}) \in \mathbb{Z}^{k-1}_{\text{prim}}$ of $z \in \mathbb{P}^{k-2}(\mathbb{Q})$.
- The intersection multiplicities of z and $\mathcal{D}_1, \ldots, \mathcal{D}_k$ at p are respectively the p-adic valuations of $z_1, \ldots, z_{k-1}, z_1 + \cdots + z_{k-1}$.
- The PSTV-A conjecture predicts $N_k(B) \sim cB^{\frac{k}{2}-1}$ (after the potential removal of a thin set).

The PSTV-A conjecture

- $N_k(B) \sim c_k B^{\frac{k}{2}-1}$ for any $k \geq 5$. (Van Valckenborgh, 2012)

The PSTV-A conjecture

- $N_k(B) \sim c_k B^{\frac{k}{2}-1}$ for any $k \geq 5$. (Van Valckenborgh, 2012)
- For $N_3(B)$ (corresponding to $X = \mathbb{P}^1_{\mathbb{Q}}$ with divisor $D = \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty]$), we only have the bounds
$$cB^{1/2} \leq N_3(B) \ll_\epsilon B^{3/5+\epsilon}.$$
(Browning, Van Valckenborgh, 2012)

The PSTV-A conjecture

- $N_k(B) \sim c_k B^{\frac{k}{2}-1}$ for any $k \geq 5$. (Van Valckenborgh, 2012)
- For $N_3(B)$ (corresponding to $X = \mathbb{P}^1_{\mathbb{Q}}$ with divisor $D = \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty]$), we only have the bounds
$$cB^{1/2} \leq N_3(B) \ll_\epsilon B^{3/5+\epsilon}.$$
(Browning, Van Valckenborgh, 2012)
- Hyperplanes in $\mathbb{P}^n$, with $\sum_{\substack{0\leq i\leq n+1 \\ i\neq j}} \frac{1}{m_i(m_i+1)} \geq 1$ for some j. (Browning, Yamagishi, 2018)

The PSTV-A conjecture

- $N_k(B) \sim c_k B^{\frac{k}{2}-1}$ for any $k \geq 5$. (Van Valckenborgh, 2012)
- For $N_3(B)$ (corresponding to $X = \mathbb{P}^1_{\mathbb{Q}}$ with divisor $D = \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty]$), we only have the bounds
$$cB^{1/2} \leq N_3(B) \ll_\epsilon B^{3/5+\epsilon}.$$
(Browning, Van Valckenborgh, 2012)
- Hyperplanes in $\mathbb{P}^n$, with $\sum_{\substack{0\leq i\leq n+1\\ i\neq j}} \frac{1}{m_i(m_i+1)} \geq 1$ for some j. (Browning, Yamagishi, 2018)
- Vector group compactifications. (PSTV-A, 2019)

The PSTV-A conjecture

- $N_k(B) \sim c_k B^{\frac{k}{2}-1}$ for any $k \geq 5$. (Van Valckenborgh, 2012)
- For $N_3(B)$ (corresponding to $X = \mathbb{P}^1_{\mathbb{Q}}$ with divisor $D = \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty]$), we only have the bounds
$$cB^{1/2} \leq N_3(B) \ll_\epsilon B^{3/5+\epsilon}.$$
(Browning, Van Valckenborgh, 2012)
- Hyperplanes in $\mathbb{P}^n$, with $\sum_{\substack{0 \leq i \leq n+1 \\ i \neq j}} \frac{1}{m_i(m_i+1)} \geq 1$ for some j. (Browning, Yamagishi, 2018)
- Vector group compactifications. (PSTV-A, 2019)
- Split toric varieties via hyperbola method. (Pieropan, Schindler, 2020)

The PSTV-A conjecture

- $N_k(B) \sim c_k B^{\frac{k}{2}-1}$ for any $k \geq 5$. (Van Valckenborgh, 2012)
- For $N_3(B)$ (corresponding to $X = \mathbb{P}^1_{\mathbb{Q}}$ with divisor $D = \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty]$), we only have the bounds
$$cB^{1/2} \leq N_3(B) \ll_\epsilon B^{3/5+\epsilon}.$$
(Browning, Van Valckenborgh, 2012)
- Hyperplanes in $\mathbb{P}^n$, with $\sum_{\substack{0 \leq i \leq n+1 \\ i \neq j}} \frac{1}{m_i(m_i+1)} \geq 1$ for some j. (Browning, Yamagishi, 2018)
- Vector group compactifications. (PSTV-A, 2019)
- Split toric varieties via hyperbola method. (Pieropan, Schindler, 2020)
- Biequivariant compactifications of the Heisenberg group. (Xiao, 2020)

The PSTV-A conjecture

- $N_k(B) \sim c_k B^{\frac{k}{2}-1}$ for any $k \geq 5$. (Van Valckenborgh, 2012)
- For $N_3(B)$ (corresponding to $X = \mathbb{P}^1_{\mathbb{Q}}$ with divisor $D = \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty]$), we only have the bounds
$$cB^{1/2} \leq N_3(B) \ll_\epsilon B^{3/5+\epsilon}.$$
(Browning, Van Valckenborgh, 2012)
- Hyperplanes in $\mathbb{P}^n$, with $\sum_{\substack{0 \leq i \leq n+1 \\ i \neq j}} \frac{1}{m_i(m_i+1)} \geq 1$ for some j. (Browning, Yamagishi, 2018)
- Vector group compactifications. (PSTV-A, 2019)
- Split toric varieties via hyperbola method. (Pieropan, Schindler, 2020)
- Biequivariant compactifications of the Heisenberg group. (Xiao, 2020)
- Powerful values of norm forms. (Streeter, 2020)

The case $k = 4$

- $N_4(B) = \#\left\{ z_1, \dots, z_4 \in \mathbb{Z}_{\neq 0} : \begin{array}{l} z_1, \dots, z_4 \text{ squareful}, |z_1|, \dots, |z_4| \leq B, \\ \gcd(z_1, \dots, z_4) = 1, z_1 + z_2 + z_3 + z_4 = 0 \end{array} \right\}.$

The case $k = 4$

- $N_4(B) = \#\left\{ z_1, \ldots, z_4 \in \mathbb{Z}_{\neq 0} : \begin{array}{l} z_1, \ldots, z_4 \text{ squareful}, |z_1|, \ldots, |z_4| \leq B, \\ \gcd(z_1, \ldots, z_4) = 1, z_1 + z_2 + z_3 + z_4 = 0 \end{array} \right\}.$
- In fact, $N_4(B) \gg B \log B$.

The case $k = 4$

- $N_4(B) = \#\left\{ z_1, \dots, z_4 \in \mathbb{Z}_{\neq 0} : \begin{array}{l} z_1, \dots, z_4 \text{ squareful}, |z_1|, \dots, |z_4| \leq B, \\ \gcd(z_1, \dots, z_4) = 1, z_1 + z_2 + z_3 + z_4 = 0 \end{array} \right\}.$
- In fact, $N_4(B) \gg B \log B$.
- For a fixed $\mathbf{y} = (y_1, \dots, y_4) \in (\mathbb{Z}_{\neq 0})^4$, let

$$N_{\mathbf{y}}(B) = \#\left\{ x_1, \dots, x_4 \in \mathbb{Z}_{\neq 0} : \begin{array}{c} y_1^3 x_1^2 + \dots + y_4^3 x_4^2 = 0, \\ \gcd(y_1^3 x_1^2, \dots, y_4^3 x_4^2) = 1, \\ |y_1^3 x_1^2|, \dots, |y_4^3 x_4^2| \leq B \end{array} \right\}.$$

Then

$$N_4(B) = \sum_{\substack{\mathbf{y} \in (\mathbb{Z}_{\neq 0})^4 \\ y_1, \dots, y_4 \text{ square-free}}} N_{\mathbf{y}}(B),$$

and

$$N_{\mathbf{y}}(B) \sim \begin{cases} c_{\mathbf{y}} B, & \text{if } y_1 \cdots y_4 \neq \square, \\ c_{\mathbf{y}} B \log B, & \text{if } y_1 \cdots y_4 = \square. \end{cases}$$

The main theorem

Define $\mathcal{T} = \# \left\{ z_1, \ldots, z_4 \in \mathbb{Z}_{\neq 0} : \begin{array}{l} z_1, \ldots, z_4 \text{ squareful}, \ \gcd(z_1, \ldots, z_4) = 1, \\ z_1 + z_2 + z_3 + z_4 = 0, \ z_1 \cdots z_4 = \square \end{array} \right\}.$

Theorem (S., 2021)

The set $\mathcal{T}$ is a thin set of Campana points. After removing it, the count $N_4(B)$ becomes

$$N(B) = cB + O_\epsilon(B^{734/735+\epsilon})$$

for an explicit constant $c > 0$. The power of B and $\log B$ in the main term agree with the PSTV-A conjecture.

$$N(B) = \sum_{\substack{\mathbf{y} \in (\mathbb{Z}_{\neq 0})^4 \\ y_1, \ldots, y_4 \text{ square-free} \\ y_1 \cdots y_4 \neq \square}} N_{\mathbf{y}}(B).$$

We need enough uniformity in our estimates for $N_{\mathbf{y}}(B)$ that we can take this sum over $\mathbf{y}$.

For any $\mathbf{a} \in (\mathbb{Z}_{\neq 0})^4$, let $N'_{\mathbf{a}}(B) = \# \left\{ \mathbf{x} \in \mathbb{N}^4 : \sum_{i=1}^4 a_i x_i^2 = 0, \max_{1 \le i \le 4} |a_i x_i^2| \le B \right\}.$

Sketch proof

Theorem (S., 2021)

Let $\mathrm{a} \in (\mathbb{Z}_{\neq 0})^4$ *and define* $A = a_1 \cdots a_4$, $\Delta = \prod_{i=1}^4 \gcd\left(a_i, \prod_{j\neq i} a_j\right)$. *If* $A \neq \square$ *and* $|A| \leq B^{4/7}$, *then*

$$N'_{\boldsymbol{a}}(B) = \frac{\mathfrak{S}_{\boldsymbol{a}}\sigma_\infty(\boldsymbol{a})B}{|A|^{1/2}} + O_\epsilon\left(\frac{B^{41/42+\epsilon}\Delta^{1/3}}{|A|^{11/24}}\right).$$

The proof uses "A new form of the circle method, and its application to quadratic forms" (Heath-Brown 1995).

$$N'_{w,\mathbf{a}}(B) = \frac{1 + O_N(B^{-N})}{B} \sum_{\mathbf{c}\in\mathbb{Z}^4} \sum_{q=1}^{\infty} q^{-4} S_{q,\mathbf{a}}(\mathbf{c}) I_{q,\mathbf{a}}(\mathbf{c}),$$

where w is a smooth weight approximating $1_{[-1,1]^4}$ and

$$S_{q,\mathbf{a}}(\mathbf{c}) = \sum_{\substack{k \bmod q \\ \gcd(k,q)=1}} \sum_{\mathrm{b} \bmod q} e_q\left(k\sum_{i=1}^4 a_i b_i^2 + \mathrm{b}\cdot\mathbf{c}\right),$$

$$I_{q,\mathbf{a}}(\mathbf{c}) = \int_{\mathbb{R}^4} w\left(\sqrt{\frac{|a_1|}{B}}x_1, \ldots, \sqrt{\frac{|a_4|}{B}}x_4\right) h\left(\frac{q}{B^{1/2}}, \frac{\sum_{i=1}^4 a_i x_i^2}{B}\right) e_q(-\mathbf{c}\cdot\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

The leading constant

$$c = \frac{1}{16} \sum_{\substack{\mathbf{y} \in (\mathbb{Z}_{\neq 0})^4 \\ y_1, \dots, y_4 \text{ square-free} \\ y_1 \cdots y_4 \neq \square}} \frac{\sigma_\infty(\mathbf{y})}{|y_1 \cdots y_4|^{3/2}} \prod_p \lim_{N \to \infty} \left(\frac{M_N(\mathbf{y}, p)}{p^{3N}} \right),$$

where

$$M_n(\mathbf{y}, p) = \# \left\{ \boldsymbol{m} \pmod{p^n} : \sum_{i=1}^4 y_i^3 m_i^2 \equiv 0 \pmod{p^n}, p \nmid m_j y_j \text{ for some } j \right\},$$

$$\sigma_\infty(\boldsymbol{y}) = \int_{-\infty}^{\infty} \int_{[-1,1]^4} e(-\theta(\mathrm{sgn}(y_1) x_1^2 + \cdots + \mathrm{sgn}(y_4) x_4^2)) \, \mathrm{d}\mathbf{x} \, \mathrm{d}\theta.$$

The constant from the PSTV-A conjecture is

$$c_{\text{PSTV-A}} = \frac{1}{16} \int_{\overline{\mathcal{U}(\mathbb{Q})}_\epsilon} H_D \, \mathrm{d}\tau_X$$
$$= \frac{1}{16} \sigma_\infty \prod_p (1 - p^{-1}) \sigma_p.$$

More thin sets

- Let $Q_{\mathbf{y}}$ be the quadric $\sum_{i=1}^{4} y_i^3 x_i^2 = 0$. Each $c_{\mathbf{y}}$ contributes a positive proportion to c, and corresponds to a thin set of Campana points.

$$\varphi_{\mathbf{y}} \colon Q_{\mathbf{y}} \to \mathbb{P}^2,$$
$$[x_0 : x_1 : x_2 : x_3] \mapsto [y_0^3 x_0^2 : y_1^3 x_1^2 : y_2^3 x_2^2].$$

The image $\varphi_{\mathbf{y}}(Q_{\mathbf{y}}(\mathbb{Q}))$ is a type II thin set of $\mathbb{P}^2(\mathbb{Q})$.

More thin sets

- Let $Q_{\mathbf{y}}$ be the quadric $\sum_{i=1}^{4} y_i^3 x_i^2 = 0$. Each $c_{\mathbf{y}}$ contributes a positive proportion to c, and corresponds to a thin set of Campana points.

$$\varphi_{\mathbf{y}} \colon Q_{\mathbf{y}} \to \mathbb{P}^2,$$
$$[x_0 : x_1 : x_2 : x_3] \mapsto [y_0^3 x_0^2 : y_1^3 x_1^2 : y_2^3 x_2^2].$$

  The image $\varphi_{\mathbf{y}}(Q_{\mathbf{y}}(\mathbb{Q}))$ is a type II thin set of $\mathbb{P}^2(\mathbb{Q})$.

- Let $\mathcal{T}_{\mathbf{y}} = \varphi_{\mathbf{y}}(Q_{\mathbf{y}}(\mathbb{Q})) \cap (\mathcal{X}, \mathcal{D})(\mathbb{Z})$. Then

$$(\mathcal{X}, \mathcal{D})(\mathbb{Z}) = \bigsqcup_{\substack{\mathbf{y} \in (\mathbb{Z}_{\neq 0})^4 \\ y_1, \ldots, y_4 \text{ square-free} \\ y_1 \cdots y_4 \neq \square}} \mathcal{T}_{\mathbf{y}}.$$

More thin sets

- Let $Q_{\mathbf{y}}$ be the quadric $\sum_{i=1}^{4} y_i^3 x_i^2 = 0$. Each $c_{\mathbf{y}}$ contributes a positive proportion to c, and corresponds to a thin set of Campana points.

$$\varphi_{\mathbf{y}} \colon Q_{\mathbf{y}} \to \mathbb{P}^2,$$
$$[x_0 : x_1 : x_2 : x_3] \mapsto [y_0^3 x_0^2 : y_1^3 x_1^2 : y_2^3 x_2^2].$$

  The image $\varphi_{\mathbf{y}}(Q_{\mathbf{y}}(\mathbb{Q}))$ is a type II thin set of $\mathbb{P}^2(\mathbb{Q})$.
- Let $\mathcal{T}_{\mathbf{y}} = \varphi_{\mathbf{y}}(Q_{\mathbf{y}}(\mathbb{Q})) \cap (\mathcal{X}, \mathcal{D})(\mathbb{Z})$. Then

$$(\mathcal{X}, \mathcal{D})(\mathbb{Z}) = \bigsqcup_{\substack{\mathbf{y} \in (\mathbb{Z}_{\neq 0})^4 \\ y_1, \dots, y_4 \text{ square-free} \\ y_1 \cdots y_4 \neq \square}} \mathcal{T}_{\mathbf{y}}.$$

- Therefore any constant in $(0, c]$ could be obtained by the removal of an appropriate thin set.

Squareful values of binary forms

- Take $X = \mathbb{P}^1_{\mathbb{Q}}$, and let H be the height on $X(\mathbb{Q})$ given by $H([x:y]) = \max(|x|,|y|)$ for $(x,y) \in \mathbb{Z}^2_{\text{prim}}$. Let D be the divisor $\frac{1}{2}V(ax^2+by^2)$, where $a \equiv b \equiv 1 \pmod 4$ and $\mu^2(ab) = 1$. Consider the orbifold (X, D) with the obvious good integral model $(\mathcal{X}, \mathcal{D})$ over $\mathbb{Z}$. The corresponding counting problem is

$$N(B) = \frac{1}{2}\#\left\{(x,y) \in \mathbb{Z}^2_{\text{prim}} : |x|,|y| \leq B, ax^2+by^2 \text{ squareful}\right\}.$$

Theorem (S., 2021)

The PSTV-A conjecture does not hold in the above setup when $a = 37, b = 109$.

$$c_{\text{PSTV-A}} = R \prod_{p \nmid 2ab} \left(1 + \frac{1+\left(\frac{-ab}{p}\right)}{(1+p^{-1})p^{3/2}}\right),$$

$$c = R \sum_{k|a} \sum_{l|b} \prod_{p \nmid 2ab} \left(\left(\frac{b}{k}\right)\left(\frac{a}{l}\right) + \frac{\left(\frac{bp}{k}\right)\left(\frac{ap}{l}\right)\left(1+\left(\frac{-ab}{p}\right)\right)}{(1+p^{-1})p^{3/2}}\right).$$

When $a = 37, b = 109$, we have $c < c_{\text{PSTV-A}}$ and so thin sets cannot explain the discrepancy between the constants.

Questions

- Under what circumstances is the prediction for the constant from the PSTV-A conjecture correct, and how can we fix it in the cases where it isn't?

Questions

- Under what circumstances is the prediction for the constant from the PSTV-A conjecture correct, and how can we fix it in the cases where it isn't?
- Can we formulate a geometric prediction for what thin sets we should remove in the PSTV-A conjecture in the spirit of Lehmann, Sengupta and Tanimoto's work for Manin's conjecture?

Questions

- Under what circumstances is the prediction for the constant from the PSTV-A conjecture correct, and how can we fix it in the cases where it isn't?
- Can we formulate a geometric prediction for what thin sets we should remove in the PSTV-A conjecture in the spirit of Lehmann, Sengupta and Tanimoto's work for Manin's conjecture?
- Is this just a feature of Campana orbifolds or can something similar happen with Peyre's constant for Manin's conjecture?