On the number of ideals which norm is a binary form of degree 3

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- $\mathcal{R}$ is a domain of $\mathbb{R}^2$.
- For $\xi > 0$ and $\mathcal{R}$ a domain of $\mathbb{R}^2$, we denote

$$\mathcal{R}(\xi) := \left\{ \boldsymbol{x} \in \mathbb{R}^2 : \frac{1}{\xi}\boldsymbol{x} \in \mathcal{R} \right\}.$$

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We look for an asymptotic estimate, when $\xi \to \infty$, of

$$Q(\xi, \mathcal{R}, F) := \sum_{\boldsymbol{x} \in \mathbb{Z}^2 \cap \mathcal{R}(\xi)} r_3(F(\boldsymbol{x})),$$

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$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

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(H3) $\forall \boldsymbol{x} \in \mathcal{R}, |F(\boldsymbol{x})| \leqslant 1$;

(H4) The form F is irreducible over $\mathbb{K}$.

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Theorem (L., work in progress)

Let $\xi > 0$, $\mathbb{K}$, χ, *F and* $\mathcal{R}$ *as before, such that (H1), (H2), (H3) and (H4) are checked.*

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$$Q(F,\xi,\mathcal{R}) = K(F)L(1,\chi)L(1,\chi^2)\mathrm{vol}(\mathcal{R})\xi^2 + O\left(\frac{\xi^2}{(\log\xi)^{0,0034}}\right) \quad (1)$$

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The constant $K_p(F)$ and $K_q(F)$ are explicit.
Furthermore, if the ring $O_{\mathbb{K}}$ is principle, we have a geometric interpretation of the constants $K_q(F)$ and $K_p(F)$ for $p \nmid q$.

Main theorem

The proof of this theorem is based on methods developed by La Bretèche and Tenenbaum.

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To prove it, we need some results on an arithmetic function called Hooley's Delta function.

The Hooley's Delta function

Definition

For $n \geqslant 1$ and $(u_1, u_2) \in \mathbb{R}^2$, we define

$$\Delta_3(n, u_1, u_2) := \sum_{\substack{d_1 d_2 | n \\ e^{u_i} < d_i \leqslant e^{u_i+1}}} 1,$$

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and

$$\Delta_3(n) := \max_{(u_1, u_2) \in \mathbb{R}^2} |\Delta_3(n, u_1, u_2)|.$$

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Theorem (Hall and Tenenbaum, '85)

For a "nice" arithmetic function g, we have

$$\sum_{n \leqslant x} g(n)\Delta_3(n) \ll x(\log x)^{o(1)}.$$

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Example

In our case, if we put $\tilde{F}(x) = F(x, 1)$, the function $\rho_{\tilde{F}}$ is a "nice" function, because $\tilde{F}$ is irreducible over $\mathbb{Q}$.

The Hooley's Delta function

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For $n \geqslant 1$, $(u_1, u_2, v_1, v_2) \in \mathbb{R}^2 \times [0,1]^2$, and f_1, f_2 two arithmetic functions, we define

$$\Delta_3(n, f_1, f_2, u_1, v_1, u_2, v_2) := \sum_{\substack{d_1 d_2 | n \\ e^{u_i} < d_i \leqslant e^{u_i + v_i}}} f_1(d_1) f_2(d_2),$$

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The Hooley's Delta function

Theorem (L., work in progress)

If χ_1 and χ_2 are two non trivial Dirichlet caracters such that $\chi_1\overline{\chi_2}$ is non trivial, then, for a "good" arithmetic function g, we have

$$\sum_{n\leqslant x} g(n)\Delta_3(n,\chi_1,\chi_2)^2 \ll x(\log x)^{\rho+o(1)},$$

where $\rho \approx 0,218$.

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where $\rho \approx 0{,}218$.

A "good" function is a positive and multiplicative function g which satisfies a prime number theorem with mean value over prime number equals to 1, but also the following equality.

$$\sum_{p\leqslant x} g(p)f(p) = O(x\,\mathrm{e}^{-c\sqrt{\log x}})$$

for $f=\chi_1$, $f=\chi_2$ and $f=\chi_1\overline{\chi_2}$.

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In our case, for $\chi_1 = \chi$, $\chi_2 = \chi^2$ and $\tilde{F}(x) = F(x, 1)$, the function $\rho_{\tilde{F}}$ is a "good" function, according to (H4).

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In our case, for $\chi_1 = \chi$, $\chi_2 = \chi^2$ and $\tilde{F}(x) = F(x,1)$, the function $\rho_{\tilde{F}}$ is a "good" function, according to (H4). Indeed, we have $\chi^3 = 1$, so $\overline{\chi^2} = \chi$.

Sketch of proof- First step : a good approximation

We use the convolution identity

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$$\begin{aligned}Q(\xi,\mathcal{R},F)&=\sum_{\boldsymbol{x}\in\mathcal{R}(\xi)}(1*\chi*\chi^2)(F(\boldsymbol{x}))\\&=\sum_{(d_1,d_2)\in(\mathbb{N}^*)^2}\chi(d_1)\chi^2(d_2)|\Lambda(d_1d_2,F)\cap\mathcal{R}(\xi)|,\end{aligned}$$

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$$\Lambda(s,F) := \{(m,n)\in\mathbb{Z}^2 \,:\, s \mid F(m,n)\}.$$

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$$\rho_F^+(s) := \sum_{\substack{1 \leqslant a,b \leqslant s \\ F(a,b) \equiv 0 \bmod s}} 1,$$

Sketch of proof- First step : a good approximation

Lemma (Daniel, 1999)

Let F a binary form of degree 3 irreducible over $\mathbb{Q}$. We have

$$\sum_{\substack{1\leqslant d_1\leqslant y_1\\ 1\leqslant d_2\leqslant y_2\\ (q,d_1d_2)=1}} \sup_{\mathcal{R}} \left| |\Lambda(d_1d_2,F)\cap\mathcal{R}(\xi)| - \mathrm{vol}(\mathcal{R})\xi^2\frac{\rho_F^+(d_1d_2)}{d_1^2d_2^2}\right| \tag{3}$$

$$\ll\left(\xi\sqrt{y_1y_2}+y_1y_2\right)\log(\xi)^{o(1)}.$$

where the supremum is taken on the set of domains $\mathcal{R}$ which check hypothesis (H1), (H2) and (H3).

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The proof of this lemma uses an approximation of the number of points of a lattice in an open convex domain, this is why we need (H1).

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The proof of this lemma uses an approximation of the number of points of a lattice in an open convex domain, this is why we need (H1). It also uses the theorem of Hall and Tenenbaum on the function Δ_3.

Sketch of proof- First step : a good approximation

Corollary

Let $y \geqslant 2$, and F as before. We have, for $u \in \mathbb{R}^+$

$$\sum_{\substack{1 \leqslant d_1 d_2 \leqslant y \\ e^u < d_2 \leqslant e^{u+1} \\ (q, d_1 d_2) = 1}} \sup_{\mathcal{R}} \left| |\Lambda(d_1 d_2, F) \cap \mathcal{R}(\xi)| - \mathrm{vol}(\mathcal{R}) \xi^2 \frac{\rho_F^+(d_1 d_2)}{d_1^2 d_2^2} \right| \tag{4}$$

$$\ll (\xi \sqrt{y} + y) \log(\xi)^{o(1)}.$$

Sketch of proof- First step : a good approximation

Corollary

Let $A > 0$ and F as before. If $2 \leqslant y \leqslant \xi^A$, we have we have

$$\sum_{\substack{1 \leqslant d_1 d_2 \leqslant y \\ (q, d_1 d_2) = 1}} \sup_{\mathcal{R}} \left| |\Lambda(d_1 d_2, F) \cap \mathcal{R}(\xi)| - \mathrm{vol}(\mathcal{R}) \xi^2 \frac{\rho_F^+(d_1 d_2)}{d_1^2 d_2^2} \right| \tag{5}$$
$$\ll_A \left(\xi\sqrt{y} + y\right) (\log \xi)^{1+o(1)}.$$

Sketch of proof- First step : a good approximation

In the equality

$$Q(\xi, \mathcal{R}, F) = \sum_{(d_1, d_2) \in (\mathbb{N}^*)^2} \chi(d_1)\chi^2(d_2)|\Lambda(d_1 d_2, F) \cap \mathcal{R}(\xi)|,$$

e can control the contribution of (d_1, d_2) which are well bounded.

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we can control the contribution of (d_1,d_2) which are well bounded. We want to make a "variable change" in the sum $(1*\chi*\chi^2)(F(\boldsymbol{x}))$, putting $d_3=F(\boldsymbol{x})/(d_1d_2)$.
For that, we need $\chi(F(\boldsymbol{x}))=1$.

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For n s.t $(n, q) = 1$,

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If $d \mid q^\infty$ then $\forall n \geqslant 1$

$$(1 * \chi * \chi^2)(dn) = (1 * \chi * \chi^2)(n).$$

Sketch of proof- Second step : Parametrization

$$Q(F,\xi,\mathcal{R}) = \sum_{d|q^\infty} \sum_{d_1|q^\infty} \sum_{\alpha\in G_1} \sum_{d_2|q^\infty} \sum_{\beta\in W_{\alpha,d_1,d_2}} Q_2\left(F_{\beta,d_1,d_2}, \frac{\xi}{d}, \mathcal{R}_{\beta,d_1,d_2}\right) \quad (6)$$

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$$G_1 := \mathrm{Ker}(\chi) \subset (\mathbb{Z}/q\mathbb{Z})^\times$$
$$Q_2(F,\xi,\mathcal{R}) := \sum_{(m,n)\in\mathcal{R}(\xi)} r_3(F(m,n))$$

Sketch of proof- Conclusion

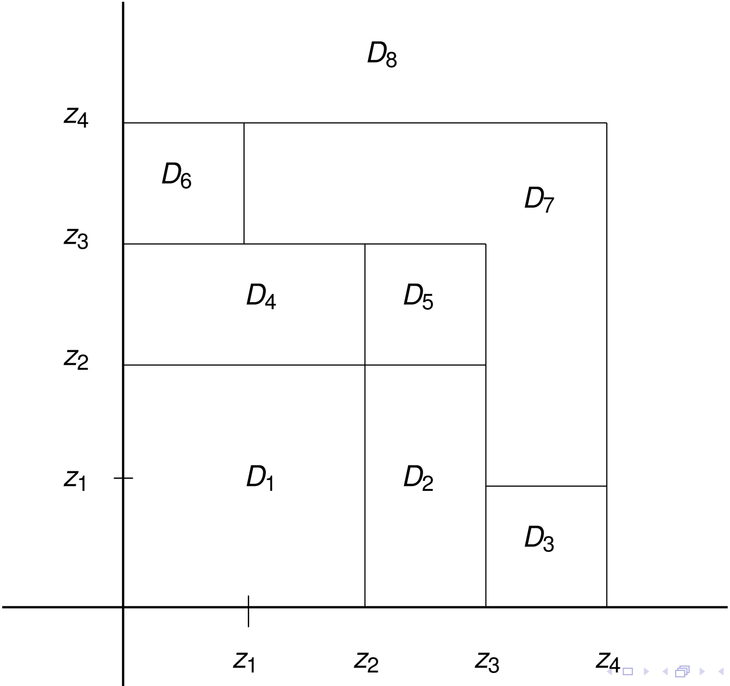
In order to estimate $Q_2(F,\xi,\mathcal{R})$, we write

$$Q_2(F,\xi,\mathcal{R}) = \sum_{(d_1 d_2,q)=1} \chi(d_1)\chi^2(d_2)|\Lambda(d_1 d_2,F)\cap\mathcal{D}_q\cap\mathcal{R}(\xi)|$$

and we put

$$\begin{aligned} z_1 &:= \xi(\log\xi)^{-4}, \\ z_2 &:= \xi(\log\xi)^{-2\delta}, \\ z_3 &:= \xi(\log\xi)^{\delta}, \\ z_4 &:= \xi(\log\xi)^{3} \end{aligned} \tag{7}$$

with $\delta = 0,0069$. Then we make the following partition on the sum over (d_1,d_2),


D_8
z_4
D_6
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Only the contributions of $(d_1, d_2) \in D_1$ and $(d_1, d_2) \in D_8$ count for the main term of the theorem. Others are error terms.

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and for all $p \in \mathcal{P}$,

$$K_p(F) = \lim_{k\to\infty} \frac{1}{p^{4k}} \left| \left\{ (\boldsymbol{x}, y, z, t) \in (\mathbb{Z}/p^k\mathbb{Z})^5 : F(\boldsymbol{x}) \equiv P(y,z,t) \bmod p^k \right\} \right|.$$

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Lemma

Let $n \in \mathbb{N}^*$ such that $(n, q) = 1$. The following properties are equivalent.

- $\chi(n) = 1$,
- *Exists $(y, z, t) \in \mathbb{Z}^3$ such that*

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for $p \mid q$. This computation uses decomposition of $pO_{\mathbb{K}}$ into prime factors and the fact that all ideals are principle.

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