Distribution of Rational Points on Toric Varieties: A Multi-Height Approach

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March 19, 2021

How to Count Rational Points?

X: an algebraic variety in $\mathbb{P}^n$ over a number field

Main Idea: We use height functions to count rational points.

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Height Machine associates counting devices to divisor classes.

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Height Machine is useful for other fields too.

Counting functions

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$$N(U(K), B) = card\{x \in U(K) : H_K(x) \leq B\}$$

Let $U \subseteq X$ be a Zariski open with some rational points.

Geometry shapes Arithmetic (for curves)

- $g = 0 \rightsquigarrow N(B) \approx cB^2$
- $g = 1 \rightsquigarrow N(B) \approx c(\log(B)^{r/2})$
- $g > 1 \rightsquigarrow N(B) \approx c$

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Another important geometric invariant is the canonical class.

Manin's Conjecture

X: Fano variety i.e. $-K_X$ is ample. Then

Conjecture

(Batryev-Manin) X: smooth projective variety. There is a finite extension F of k with $X(F)$ Zariski dense in X. Moreover, for a small enough open set U

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This is unfortunately false in general:

A Counter-Example

Example

(Batyrev-Tschinkel) X_{n+2}: hypersurface in $\mathbb{P}^n \times \mathbb{P}^3$ given by

$$\ell_0(\underline{x})y_0^3 + \ell_1(\underline{x})y_1^3 + \ell_2(\underline{x})y_2^3 + \ell_3(\underline{x})y_3^3 = 0$$

where $n > 2$

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3. $\mathrm{P}ic(X_{n+2}) \cong \mathbb{Z}^2$
4. $\mathbb{Q}(\sqrt{-3}) \subseteq k_0$ a number field

$$N(U(k_0), B) \geq cB(\log(B))^3$$

Some Good News

Some Examples

MC is true for some classes of varieties including:

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MC is true for some classes of varieties including:

- Toric Varieties
- Some equivariant compactifications
- Flag varieties
- Smooth complete intersections of small degree

Accumulating Subvarieties

Let U be a countable set, $h_L : U \to \mathbb{R}_+$ a function such that its growth is finite for all bounds.

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The Connection with the growth function:

$$\beta = \limsup_{B \to \infty} \frac{\log \ N(L,U,B)}{\log \ B}$$

Accumulating subvarieties: $X \subset U$ is **accumulating** when

$$\beta_U(L) = \beta_X(L) > \beta_{U \setminus X}(L)$$

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4. Ample divisors D are of the form $D = aL - bE$ for $a > b > 0$.

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$$N(E(\mathbb{Q}), aL - bE, B) = \begin{cases} B^{(m+1)/(a-b)} & \text{if } \frac{m+1}{a-b} > \frac{n-m}{b} \\ B^{(m+1)/(a-b)} \log B & \text{if } \frac{m+1}{a-b} = \frac{n-m}{b} \\ B^{(n-m)/b} & \text{if } \frac{m+1}{a-b} < \frac{n-m}{b} \end{cases}$$

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Corollary: The exceptional curve of the blowing up of $\mathbb{P}^2$ at a point is accumulating for the anticanonical divisor $(3, 1)$.

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2. Universal torsors by Salberger, de la Bretèche, et al.

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Our method is based on harmonic analysis.
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e.g. An orthogonal group of a quadratic form non-vanishing $/ \mathbb{Q}$.

Fans and toric varieties

Split Tori

Data of a Fan:

1. M - free abelian group of finite rank,
   $N := \mathrm{Hom}(M, \mathbb{Z})$ - the dual abelian group,
   $N_{\mathbb{R}} := N \otimes \mathbb{R}$
2. a fan $\Sigma = \{\sigma\}$ is a finite collection of cones in $N_{\mathbb{R}}$:
   - $0 \in \Sigma$
   - $\forall \sigma \in \Sigma$, every face $\tau \subset \sigma$ is in Σ
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Regular Fans: if the generators of every $\sigma \in \Sigma$ form part of a basis of N.

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Toric Varieties:

$$X_\Sigma := \bigcup_{\sigma \in \Sigma} U_\sigma \quad \text{where} \quad U_\sigma := \mathrm{Spec}(\overline{K}[M \cap \check{\sigma}])$$

Toric Varieties: An Example and Properties

Example

Input: Σ: a d-dimensional fan with

1. 1-dimensional cones generated by

$$e_1, \ldots, e_d, e_{d+1}$$

   where $e_{d+1} = -\sum e_i$.

2. m-dimensional cones generated by m-subsets of them.

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- 1-dimensional generators $e_1, \ldots, e_n$ of Σ correspond to T-invariant boundary divisors $D_1, \ldots, D_n$ - irreducible components of $X_\Sigma \setminus \mathrm{T}$
- Σ regular $\Rightarrow$ X_Σ smooth.

Picard group and Heights

$PL(\Sigma)$ - piecewise linear $\mathbb{Z}$-valued functions ϕ on Σ determined by $\{m_{\sigma,\phi}\}_{\sigma\in\Sigma}$, i.e., by its values $\phi(e_j)$, $j = 1, \ldots, n$.

$$0 \to M \to PL(\Sigma) \xrightarrow{\pi} \mathrm{Pic}(X_\Sigma) \to 0$$

- every divisor is equivalent to a linear combination of boundary divisors $D_1, \ldots, D_n$, and ϕ is determined by its values on $e_1, \ldots, e_n$ and these are denoted as s_j.
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Height functions: Global heights are product of local heights:

1. Non-archimedean case:

$$H_{\Sigma,v}(x_v, \phi) = exp(\phi(\bar{x}_v) \log q_v)$$

2. Archimedean case:

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$X = \mathbb{P}^1$, $\mathrm{P}ic^{\mathsf{T}}(\mathbb{P}^1) =< [0], [\infty] >$, $\mathsf{s} = (2, 3)$.

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$$H(x) = \prod_v H_v(x)$$

Height zeta function:

$$\mathcal{Z}_\Sigma(\phi) := \sum_{x \in \mathrm{T}(K)} H_\Sigma(x,\phi)^{-1}$$

Poisson Formula (Anisotropic T):

$$\mathcal{Z}_\Sigma(\phi) = \frac{1}{b_S(T)} \sum_{\chi \in (T(\mathbb{A}_K)/T(K))^*} \hat{H}_\Sigma(\chi,-\phi)$$

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Proposition (Batryev and Tschinkel)

$\mathcal{Z}_{\Sigma}(\phi)$

1. Holomorphic for $\Re(s_j) > 1$
2. Has a meromorphic extension to the domain $\Re(s_j) > 1 - \delta$ with poles of order ≤ 1 along $s_j = 1$

Multi-Height Variant of the Batryev-Tschinkel Theorem

Restricting $\mathcal{Z}_\Sigma(\phi)$ to $\mathcal{Z}_{\Sigma,L}(s) \rightsquigarrow$ verification of Manin's Conjecture for Toric Varieties by B and T.

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Why useful?
This may remove the influence of accumulating subvarieties in some cases.

$\rightsquigarrow$Apply Peyre's multiheights approach for toric varieties.

Multi-Height Variant of the Batryev-Tschinkel Theorem II

Consider the behaviour of

$$N := card\{x \in X(K) : H_i(x) \leq B^{\beta_i} \text{ for } \ i = 1, \ldots, m\}$$

as $B \to \infty$ where H_i's are metrized heights determined by T-invariant D_i's yielding the ample divisor

$$\sum_i s_i D_i$$

Then we claim the following:

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Then we claim the following:

Main Theorem (Demirhan and Takloo-Bighash)

There are positive real constants P_β and θ such that

$$N = N(B^\beta) = B^{\beta_1+\beta_2+\ldots+\beta_m}(P_\beta + \mathcal{O}(B^{-\theta}))$$

as $B \to \infty$

Sketch of the proof: Reduction Argument

It suffices to prove the theorem for X_Δ replaced by T.

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Key point: Any T-invariant divisor is itself a toric variety with a fan that is completely explicitly determined from the fan of the original toric variety.

This allows us to do an inductive argument on the dimension of the toric variety.

Sketch of the proof: Reduction Argument (cont.)

Consider

$$D_1 = T_1 \cup (D_1 \setminus T_1)$$

Sketch of the proof: Reduction Argument (cont.)

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Replace the data

$$(X_\Delta, T_1, D_1 \ldots, D_r)$$

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Assume we show that

$$\left|\{p \in T(F) : H_{D_1}(P) \leq B^{a_1}, \ldots, H_{D_r}(P) \leq B^{a_r}\}\right| = CB^{a_1+a_2+\ldots+a_r} + \mathcal{O}(B^{a_1+a_2+\ldots+a_r-\epsilon})$$

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Sketch of the proof: Reduction Argument (cont.)

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$\therefore$ The problem of counting is reduced to counting rational points in a general tori.

Sketch of the proof: Anisotropic Case

We consider the arithmetic function f

$$
\begin{aligned}
f : C &\longrightarrow \mathbb{N} \\
(c_1, c_2, \dots, c_m) &\longmapsto card\{x \in X(K) : H_i(x) = c_i \text{ for each i, } 1 \leq i \leq m\}
\end{aligned}
$$

then define

Multi Height Zeta Function

$$F(\mathrm{s}) := \sum_{c_1 \in C} \cdots \sum_{c_m \in C} \frac{f(c_1, \dots, c_m)}{c_1^{s_1} \dots c_m^{s_m}} \tag{1}$$

Then F satisfies nice properties:

Sketch of the proof: Anisotropic Case (cont.)

1. $F(\mathbf{s})$ is absolutely convergent for $\Re(\mathbf{s}) > \mathbf{1}$ where $\mathbf{1} = (1, \ldots, 1)$

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- $F(\mathbf{s}) \rightsquigarrow$ a sum of the terms $\frac{1}{\prod_{1 \le i \le m} H_i(x)^{s_i}}$ up to $\mathcal{O}(1)$

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1. $F(\mathrm{s})$ is absolutely convergent for $\Re(\mathrm{s}) > 1$ where $1 = (1, \ldots, 1)$

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- D_i's are T-invariant divisors yielding a very ample divisor and s_i's are in $\mathrm{Pic}(X)_{\mathbb{C}}$ Taking the sum of both sides we obtain:

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- D_i's are T-invariant divisors yielding a very ample divisor and s_i's are in $\mathrm{Pic}(X)_{\mathbb{C}}$ Taking the sum of both sides we obtain:

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So $F(\mathrm{s})$ is absolutely convergent for $\Re(\mathrm{s}) > 1$. $\square$

Sketch of the proof: Anisotropic Case (cont.)

2. Consider $H(\mathrm{s}) := F(\mathrm{s}+1)\prod_{i=1}^{m}\pi^{(i)}(\mathrm{s})$

Sketch of the proof: Anisotropic Case (cont.)

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Proof.

$F(\mathrm{s}) = Z_{\Sigma}(\phi)$ and so by a Theorem in Batyrev and Tschinkel the claim follows. □

Sketch of the proof: Anisotropic Case (cont.)

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Sketch of the proof: Anisotropic Case (cont.)

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$$|H(\mathsf{s})| =$$
$$\mathcal{O}\left(\prod_{i=1}^{n}(|\Im(\pi^{(i)}(\mathsf{s})| + 1)^{1-\frac{1}{3}min(0,\Re(\pi^{(i)(\mathsf{s})}))}) * \left(1 + ||\Im(\mathsf{s})_1^{\epsilon}||\right)\right)$$

Sketch of the proof: Anisotropic Case (cont.)

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$$\mathcal{O}\left(\prod_{i=1}^{n} (|\Im(\pi^{(i)}(s)| + 1)^{1-\frac{1}{3} min(0, \Re(\pi^{(i)(s)}))}) * (1 + ||\Im(s)_1^{\epsilon}||) \right)$$

is uniformly valid in the region $D(1/2m - \epsilon')$ when $Re(s) < 1$

Sketch of the proof: Anisotropic Case (cont.)

In the anisotropic case the Poisson summation formula applied to the height zeta function gives a discrete sum over a lattice of characters.

Sketch of the proof: Anisotropic Case (cont.)

In the anisotropic case the Poisson summation formula applied to the height zeta function gives a discrete sum over a lattice of characters.

The estimates in this case follow from standard bounds for L-functions and integration by parts as described in the paper Batyrev and Tschinkel on anisotropic tori.

A Tauberian Theorem of R. de la Bretèche

Theorem (R. de la Bretèche)

f: arithmetic function on $\mathbb{N}^m$, F: its Dirichlet Series:

$$F(\mathrm{s}) = \sum_{1 \leq d_1}^{\infty} \cdots \sum_{1 \leq d_m}^{\infty} \frac{f(d_1 \ldots, d_m)}{d_1^{s_1} \ldots d_m^{s_m}}$$

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with the following properties:

1. $F(\mathrm{s})$ is absolutely convergent for s with $\Re(\mathrm{s}) > \alpha$ where $\alpha \in (\mathbb{R}^+)^m$
2. There exists a family $\mathscr{L}$ of non trivial linear forms ℓ_i where $1 \leq i \leq n$ and another finite such family h^r (not necessarily non-trivial) in $\mathcal{L}\mathbb{R}_m^+(\mathbb{C})$ such that

$$H(\mathrm{s}) := F(\mathrm{s} + \alpha) \prod_{i=1}^{n} \ell^{(i)}(\mathrm{s})$$

A Tauberian Theorem of R. de la Bretèche (cont.)

Theorem (R. de la Bretèche) cont.

can be extended to a holomorphic function in the domain

$$\mathcal{D}(\delta_1, \delta_3) := \{\mathsf{s} \in \mathbb{C}^m : \Re(\ell^{(i)}(\mathsf{s})) > -\delta_1, \Re(h^{(r)(\mathsf{s})}) > -\delta_3 \ \forall i, r\}$$

for some $\delta_1, \delta_3 > 0$

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Then

$$\sum_{d_1 \leq X^{\beta_1}} \cdots \sum_{d_m \leq X^{\beta_m}} f(d_1 \ldots, d_m) = X^{<\alpha, \beta>}(Q_\beta(\log\ X) + \mathcal{O}(X^{-\theta}))$$

where $Q_\beta \in \mathbb{R}[X]$ and $\theta > 0$

Technical Set up: Sufficient Functions

Definition

$E := \mathbb{R}^p$, $E_{\mathbb{C}}$: its complexification, V: a subspace in E. A non-negative real valued function c on V is called **sufficient** if

1. For any subspace $U \subset V$ and $v \in V$, the function $U \to \mathbb{R}$ defined by $u \mapsto c(v+u)$ is measurable on U and
$$c_U(v) := \int_U c(v+u)du$$
is finite.
2. For any subspace $U \subset V$ and for each $v \in V \setminus U$ we have
$$\lim_{\tau \to \pm\infty} c_U(\tau \cdot v) = 0$$

Technical Set up: Distinguished Functions

$\ell_1, \ldots, \ell_m$: linearly independent linear forms on E,
B: convex and open neighborhood of $0 = (0, \ldots, 0)$ in E such that $\ell_j(x) > -1$,
$T_B := B + iE$, $f \in \mathcal{M}(T_B)$

Technical Set up: Distinguished Functions

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$T_B := B + iE$, $f \in \mathcal{M}(T_B)$

Definition

f is called as **distinguished** with respect to the data consisting of V and linearly independent linear forms $\ell_1, \dots, \ell_m$ if it satisfies

1. The function
$$g(\mathrm{s}) := f(\mathrm{s}) \prod_{j=1}^{m} \frac{\ell_j(\mathrm{s})}{\ell_j(\mathrm{s}) + 1}$$
is holomorphic in T_B.
2. For some c for all compact subsets $K \subset T_B$, there is a constant κ with
$$|g(\mathrm{s} + iv)| \leq \kappa c(v)$$

A Crucial Property of Distinguished Functions

Let C be a connected component of $B \setminus \cup Ker(\ell_j)$. On T_C define:

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Let C be a connected component of $B \setminus \cup Ker(\ell_j)$. On T_C define:

$$\tilde{f}_C(\mathrm{s}) := \frac{1}{(2\pi)^d} \int_V f(\mathrm{s} + iv)dv$$

Proposition (Strauch and Tschinkel)

Let f be distinguished function w.r.t some data. Then

1. $\tilde{f}_C : T_C \to \mathbb{C}$ is holomorphic.
2. There is an open and convex neighborhood $\tilde{B}$ of $0 = (0, \dots, 0)$, containing C, and linear forms $\widetilde{\ell_i}$ $(1 \leq i \leq \tilde{m})$ vanishing on V such that

$$s \mapsto \tilde{f}_C(s) \prod_{j=1}^{\tilde{m}} \tilde{\ell}_j(s)$$

has a holomorphic continuation to $T_{\tilde{B}}$

Idea of the proof: General Case

Anisotropic Case $\rightsquigarrow$ Base Case.

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General Case:
Consider: $\tilde{f}_C(s) := \frac{1}{(2\pi)^d} \int\limits_V f(s+iv)\, dv$ where $d = \; dim\; V$

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By using the method of Strauch and Tschinkel (1999), we do Induction on d.

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